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Ordered Integer Quadrilaterals with a Fixed Perimeter

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Abstract : Given a positive integer n, how many quadrilaterals are there with ordered integer sides and perimeter n? Denoting the number of such quadrilaterals by $Q(n)$, the answer is given by:

$$Q(n) = \left\{\frac{1}{576} n\,(n+3)\,(2n+3) - \frac{(-1)^n}{192} n\,(n-5)\right\}.$$

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1 Introduction

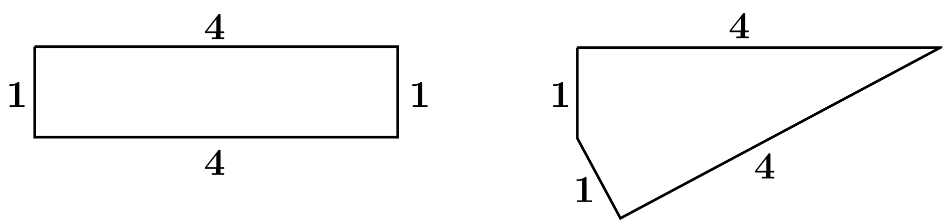
The issue of counting triangles with integer sides came to our mind for the first time in the context of teaching "Combinatorics and its Applications" course to master's graduate students, and we were not aware that the problem had been raised in the past and was finally answered by proposing the analytical formula, reference [6] shows that:

$$T(n) = \left\{ \frac{n^2}{12} \right\} - \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n+2}{4} \right\rfloor,$$

where $\{x\}$ is the nearest integer function and $\lfloor x \rfloor$ the greatest integer function.

For our part, we tried to search the answer using our available means. The natural approach from our point of view was to use the integer partition theory, and the result was the same (see [5]).

In the following, we will deal with the same problem, but by regarding the number $Q(n)$ corresponding to the non-congruent quadrilaterals with integer sides and perimeter n, which have the sequence of their sides ordered, which we will just call ordered integer quadrilaterals. For example, the 4-tuple $(1, 1, 4, 4)$ of perimeter $n = 10$ is ordered; it can be rearranged to generate an unordered 4-tuple $(1, 4, 1, 4)$, so that the first forms a kite and the second a rectangle as shown in the figure below:



2 Preliminary results

The partition of $n \in \mathbb{N}$ into k parts is a tuple $\pi = (\pi_1, \ldots, \pi_k) \in \mathbb{N}^k, k \in \mathbb{N}$, such that:

$$n = \pi_1 + \cdots + \pi_k, \ 1 \leq \pi_1 \leq \cdots \leq \pi_k,$$

where the nonnegative integers π_i are called parts. We denote the number of partitions of n into k parts by $p(n, k)$.

Lemma 1 *For $n \geq 4$, we have:*

$$Q(n) = p(n, 4) - \sum_{m=3}^{\lfloor \frac{n}{2} \rfloor} p(m, 3) \cdot$$

Proof. At first sight, it should be noted that any partition of n into four parts generates an ordered integer quadrilateral and vice versa, except the partitions for which the sum of its three small parts does not exceed the largest part, due to the triangle inequality, the such partitions verify:

$$n = a + b + c + d, \quad 1 \le a \le b \le c \le d \quad \text{and} \quad a + b + c \le d,$$

or

$$n - d = a + b + c, \quad 1 \le a \le b \le c \le d \le n.$$

But

$$n - d \le d \Longleftrightarrow n - d \le \frac{n}{2}\cdot$$

Hence

$$Q(n) = p\,(n, 4) - \sum_{m=3}^{\left\lfloor \frac{n}{2} \right\rfloor} p\,(m, 3)\cdot$$

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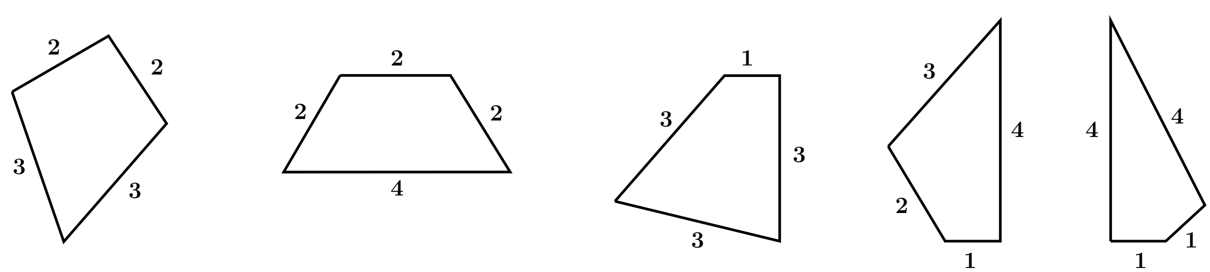
If we consider, for example, the perimeter $n = 10$, then the number of partitions of n is 9, which are: 7111, 6211, 5311, <u>4411</u>, 5221, <u>4321</u>, <u>3331</u>, <u>4222</u> and <u>3322</u>, they form a quadrilateral only those we have underlined as shown in the figure below:



As we can check:

$$p\,(10, 4) - \sum_{m=3}^{5} p(m, 3) = 9 - (1 + 1 + 2) = 5.$$

It should be noted that each quadrilaterals in the figure above represents an equivalence class of quadrilaterals that all share the same partition. So, the number $Q(n)$ counts only the non-congruent ordered integer quadrilaterals representing the equivalence classes modulo the same partition.

Lemma 2 *For $n \ge 3$, we get:*

$$\sum_{m=3}^{n} p\,(m, 3) = \frac{n\,(n-2)\,(2n+7)}{72} + \frac{1}{3}\left\lfloor \frac{n}{3} \right\rfloor + \frac{1 - (-1)^n}{16}\cdot$$

Proof. Let $f(z)$ be the known generating function of $p(m,3)$ [5]:

$$f\left(z\right)=\frac{z^{3}}{\left(1-z\right)\left(1-z^{2}\right)\left(1-z^{3}\right)}.$$

Then

$$\sum_{m=0}^{n}p(m,3)=[z^{n}]\left(\frac{f\left(z\right)}{1-z}\right).$$

From expanding $\frac{f\left(z\right)}{1-z}$ in partial fractions, we obtain:

$$\frac{f\left(z\right)}{1-z}=\frac{1}{36}\frac{z^{3}+4z^{2}+z}{\left(1-z\right)^{4}}+\frac{1}{24}\frac{z^{2}+z}{\left(1-z\right)^{3}}-\frac{1}{12}\frac{z}{\left(1-z\right)^{2}}-\frac{1}{17}\frac{1}{1-z}-\frac{1}{16}\frac{1}{1+z}+\frac{1}{9}\frac{1+z}{1+z+z^{2}}.$$

Since

$$\begin{aligned}
\frac{1}{1-z} &= \sum_{n\geq 0}z^{n},\\
\frac{1}{1+z} &= \sum_{n\geq 0}\left(-1\right)^{n}z^{n},\\
\frac{z}{\left(1-z\right)^{2}} &= \sum_{n\geq 0}nz^{n},\\
\frac{z^{2}+z}{\left(1-z\right)^{3}} &= \sum_{n\geq 0}n^{2}z^{n},\\
\frac{z^{3}+4z^{2}+z}{\left(1-z\right)^{4}} &= \sum_{n\geq 0}n^{3}z^{n},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1+z}{1+z+z^{2}} &= \frac{1-z^{2}}{1-z^{3}},\\
&= \frac{1}{1-z^{3}}-x^{2}\frac{1}{1-z^{3}},\\
&= \sum_{n\geq 0}z^{3n}-\sum_{n\geq 0}z^{3n+2},\\
&= \sum_{n\geq 0}a_{n}z^{n},
\end{aligned}$$

where

$$a_{n}=\begin{cases} 1, & n\equiv 0 \pmod 3,\\ 0, & n\equiv 1 \pmod 3,\\ -1, & n\equiv 2 \pmod 3.\end{cases}$$

In a simplified way,

$$a_{n}=1-n+3\left\lfloor\frac{n}{3}\right\rfloor.$$

Summing all coefficients of z^n, the result yields.

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Corollary 3 *For $n \geq 6$, we have:*

$$\sum_{m=3}^{\left\lfloor \frac{n}{3} \right\rfloor} p\left(n,3\right) = \frac{1}{576}\left(2n^3+3n^2-59n+30\right)+\frac{\left(-1\right)^n}{192}\left(n^2+n-10\right)+\frac{1}{3}\left\lfloor \frac{n}{6} \right\rfloor+\frac{1-\left(-1\right)^{\left\lfloor \frac{n}{2} \right\rfloor}}{16}\cdot$$

Proof. While observing that

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}-\frac{1-\left(-1\right)^n}{4},$$

we get, from Lemma 2,

$$\frac{1}{72}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor \left(2\left\lfloor \frac{n}{2} \right\rfloor+7\right) = \frac{1}{576}\left(2n^3+3n^2-59n+30\right)+\frac{\left(-1\right)^n}{192}\left(n^2+n-10\right)\cdot$$

Hence, the result follows.

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3 Main result

Theorem 4 *For $n \geq 4$, we have:*

$$Q(n) = \left\{\frac{1}{576}n\left(n+3\right)\left(2n+3\right)-\frac{\left(-1\right)^n}{192}n\left(n-5\right)\right\}\cdot$$

Proof. Reference [3] shows that the generating function of $p(n,4)$ is as follows:

$$g\left(z\right) = \frac{z^4}{\left(1-z\right)\left(1-z^2\right)\left(1-z^3\right)\left(1-z^4\right)}\cdot$$

Via straightforward calculations, it can be proved that

$$p\left(n,4\right) = \frac{n^3}{144}+\frac{n^2}{48}-\frac{\left(1-\left(-1\right)^n\right)n}{32}+\frac{\left(-1\right)^n}{32}-\frac{13}{288}+\frac{\alpha_n}{72},$$

where

$$\alpha_n \in \left\{-17,-9,-8,-1,0,1,8,9,17\right\}\cdot$$

Then, from Lemma 1 and Corollary 3, we get:

$$Q_n = \frac{1}{576} n (n+3)(2n+3) - \frac{(-1)^n}{192} n (n-5) + \beta_n,$$

where

$$\beta_n = -\frac{23}{144} + \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{16} + \frac{1}{3}\left(\frac{n}{6} - \left\lfloor \frac{n}{6} \right\rfloor\right) + \frac{(-1)^n}{12} - \frac{\alpha_n}{72},$$

with

$$\beta_n \in \left\{-\frac{3}{8}, -\frac{1}{4}, -\frac{11}{72}, -\frac{5}{36}, -\frac{1}{8}, -\frac{1}{36}, -\frac{1}{72}, 0, \frac{5}{72}, \frac{7}{72}, \frac{2}{9}, \frac{4}{9}\right\}.$$

Since $Q(n)$ is an integer and $|\beta(n)| < 1/2$, we finally get:

$$Q(n) = \left\{\frac{1}{576} n (n+3)(2n+3) - \frac{(-1)^n}{192} n (n-5)\right\}.$$

■

By using a computer algebra package, Theorem 4 allows us to obtain $Q(n)$ for large some values of n. The following table is introduced to illustrate a few:

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Q(n))$	1	1	1	2	3	4	5	7	8	11	12	16	18	23	24	31	33

4 Concluding remark

The values $Q(n)$ in the table above are sequence A062890 in the Online Encyclopedia of Integer Sequences [7], but to our knowledge, no formula has been given explicitly for this sequence.

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