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Packing Coloring of Undirected and Oriented Generalized Theta Graphs

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Abstract: The packing chromatic number $\chi_\rho(G)$ of an undirected (resp. oriented) graph G is the smallest integer k such that its set of vertices $V(G)$ can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, in such a way that every two distinct vertices in V_i are at distance (resp. directed distance) greater than i in G for every i, $1 \le i \le k$. The generalized theta graph $\Theta_{\ell_1,\ldots,\ell_p}$ consists in two end-vertices joined by $p \ge 2$ internally vertex-disjoint paths with respective lengths $1 \le \ell_1 \le \cdots \le \ell_p$.
We prove that the packing chromatic number of any undirected generalized theta graph lies between 3 and $\max\{5, n_3 + 2\}$, where $n_3 = |\{i \ / \ 1 \le i \le p,\ \ell_i = 3\}|$, and that both these bounds are tight. We then characterize undirected generalized theta graphs with packing chromatic number k for every $k \ge 3$. We also prove that the packing chromatic number of any oriented generalized theta graph lies between 2 and 5 and that both these bounds are tight.

Keywords: Packing coloring; Packing chromatic number; Theta graph; Generalized theta graph.

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1 Introduction

All the graphs we considered are simple and loopless. For an undirected graph G, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. The *distance* $d_G(u,v)$, or simply $d(u,v)$ when G is clear from the context, between vertices u and v in G is the length (number of edges) of a shortest path joining u and v. The *diameter* of G is the maximum distance between two vertices of G. We denote by P_n, $n \geq 1$, the path of order n and by C_n, $n \geq 3$, the cycle of order n.

A *packing k-coloring* of G is a mapping $\pi : V(G) \to \{1,\dots,k\}$ such that, for every two distinct vertices u and v, $\pi(u) = \pi(v) = i$ implies $d(u,v) > i$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a packing k-coloring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets $V_1, \dots, V_k$, in such a way that every two vertices in V_i are at distance greater than i in G for every i, $1 \leq i \leq k$.

This notion extends to digraphs in a natural way [15], by considering the *(weak) directed distance* between vertices, defined as the number of arcs in a shortest directed path linking these vertices, in either direction.

Packing coloring of undirected graphs has been introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [12, 13] under the name *broadcast coloring* and has been studied by several authors in recent years. Several papers deal with the packing chromatic number of certain classes of undirected graphs such as trees [3, 4, 13, 16, 17], lattices [4, 5, 9, 10, 14, 18], Cartesian products [4, 9, 16], distance graphs [6, 7, 19] or hypercubes [13, 20, 21]. Complexity issues of the packing coloring problem were adressed in [1, 2, 3, 8, 11, 13].

Let H be a subgraph of G. Since $d_G(u,v) \leq d_H(u,v)$ for any two vertices $u, v \in V(H)$, the restriction to $V(H)$ of any packing coloring of G is a packing coloring of H. This property obviously holds for digraphs as well. Hence, having packing chromatic number at most k is a hereditary property:

Proposition 1 *Let G and H be two undirected graphs, or two digraphs. If H is a subgraph of G, then $\chi_\rho(H) \leq \chi_\rho(G)$.*

Fiala and Golovach [8] proved that determining the packing chromatic number is an NP-hard problem for undirected trees. The exact value of the packing chromatic number of undirected trees with diameter at most 4 was given in [13]. The packing chromatic number of undirected paths and cycles has been determined by Goddard *et al.*:

Theorem 2 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [13])

1. *For every $n \geq 1$, $\chi_\rho(P_n) \leq 3$. Moreover, $\chi_\rho(P_n) = 1$ if and only if $n = 1$ and $\chi_\rho(P_n) = 2$ if and only if $n \in \{2, 3\}$.*
2. *For every $n \geq 3$, $3 \leq \chi_\rho(C_n) \leq 4$. Moreover, $\chi_\rho(C_n) = 3$ if and only if $n = 3$ or $n \equiv 0 \pmod 4$.*

In this paper, we consider undirected graphs and *orientations* of undirected graphs, obtained by giving to each edge of such a graph one of its two possible orientations. The so-obtained *oriented graphs* are thus digraphs having no pair of opposite arcs. Let G be any orientation of an undirected graph G. Since for any two vertices u, v in $V(G)$ we have $d_G(u, v) \leq d_G(u, v)$, where $d_G(u, v)$ denotes the directed distance between u and v, we get:

Proposition 3 *For every orientation G of an undirected graph G, $\chi_\rho(G) \leq \chi_\rho(G)$.*

Let u be a vertex in an oriented graph G. We say that u is a *source* if u has no incoming arc and that u is a *sink* if u has no outgoing arc. If uvw is a directed path in G, then $d_G(u, w) \leq 2$. Hence, u and w cannot be both assigned color 2 in any packing coloring of G. From this observation, we get an easy characterization of oriented graphs with packing chromatic number 2:

Proposition 4 (Laïche, Bouchemakh and Sopena [15])
For every orientation G of an undirected graph G, $\chi_\rho(G) = 2$ if and only if (i) G is bipartite and (ii) one part of the bipartition of G contains only sources or sinks in G.

In [15], we determined the packing chromatic number of undirected and oriented generalized coronae of paths and cycles. In particular, the packing chromatic number of oriented paths and cycles is given as follows:

Theorem 5 (Laïche, Bouchemakh and Sopena [15])
Let P_n be any orientation of the path P_n. Then, for every $n \geq 2$, $2 \leq \chi_\rho(P_n) \leq 3$. Moreover, $\chi_\rho(P_n) = 2$ if and only if one part of the bipartition of P_n contains only sources or sinks in P_n.

Theorem 6 (Laïche, Bouchemakh and Sopena [15])
Let C_n be any orientation of the cycle C_n. Then, for every $n \geq 3$, $2 \leq \chi_\rho(C_n) \leq 4$. Moreover, $\chi_\rho(C_n) = 2$ if and only if C_n is bipartite and one part of the bipartition contains only sources or sinks in C_n, and $\chi_\rho(C_n) = 4$ if and only if C_n is a directed cycle (all arcs have the same direction), $n \geq 5$ and $n \not\equiv 0 \pmod 4$.

The *generalized theta graph* $\Theta_{\ell_1,\dots,\ell_p}$ is the graph obtained by identifying the end-vertices of $p \geq 2$ paths with respective lengths $1 \leq \ell_1 \leq \cdots \leq \ell_p$. (Since we only consider simple graphs, note here that we necessarily have $\ell_2 \geq 2$.) Packing colorings of undirected generalized theta graphs were considered by William and Roy in [22] who gave some necessary condition for such a graph to have packing chromatic number 4. In this paper, we determine the packing chromatic number of every undirected generalized theta graph.

Our paper is organized as follows. In Section 2 we provide tight lower and upper bounds on the packing chromatic number of undirected generalized theta graphs and characterize undirected generalized theta graphs with any given packing chromatic number. In Section 3, we provide tight lower and upper bounds on the packing chromatic number of oriented generalized theta graphs.

2 Undirected generalized theta graphs

In this section, we determine the packing chromatic number of undirected generalized theta graphs $\Theta_{\ell_1,\dots,\ell_p}$. Since we only consider undirected graphs in this section, we will simply write generalized theta graph instead of undirected generalized theta graph.

In the rest of this paper, we denote by u and v the end-vertices of the theta graph $\Theta_{\ell_1,\dots,\ell_p}$ and by $P_i = ux_i^1 \dots x_i^{\ell_i-1}v$ the corresponding paths of length ℓ_i for every i, $1 \leq i \leq p$. Moreover, we denote by n_ℓ, $\ell \geq 1$, the number of paths of length ℓ, that is

$$n_\ell = |\{i \ / \ 1 \leq i \leq p,\ \ell_i = \ell\}|.$$

In order to describe k-colorings of paths, we use *color patterns*, given as words on the alphabet $\{1,\dots,k\}$, using standard notation from Combinatorics on Words, with $u^+ = u^*u$ for every word u. Hence, for instance, the color pattern $12(1312)^*4$ describes colorings of the form 124, 1213124, 1213121312 . . . 4.

We first prove the following general upper bound:

Theorem 7 *For every generalized theta graph $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 2$,*

$$\chi_\rho(\Theta) \leq \max\{5, n_3 + 2\}.$$

Moreover, this upper bound is tight whenever $n_3 \geq 3$.

Proof. We first prove that $\chi_\rho(\Theta) \leq 5$ whenever $n_3 \leq 3$. Let $\varphi : V(\Theta) \longrightarrow \{1,\dots,5\}$ be the mapping defined as follows:

1. $\varphi(u) = 4$, $\varphi(v) = 5$,

2. the (at most 3) paths of length 3 are colored using the distinct patterns 4125, 4215 and 4315,
3. if $\ell_i \equiv 0 \pmod 4$, $\ell_i \geq 4$, $\varphi(P_i)$ is defined by the pattern $4121(3121)^*5$,
4. if $\ell_i \equiv 1 \pmod 4$, $\ell_i \geq 5$, $\varphi(P_i)$ is defined by the pattern $41231(2131)^*5$,
5. if $\ell_i \equiv 2 \pmod 4$, $\varphi(P_i)$ is defined by the pattern $41(2131)^*5$,
6. if $\ell_i \equiv 3 \pmod 4$, $\ell_i \neq 3$, $\varphi(P_i)$ is defined by the pattern $412(3121)^*5$.

We claim that φ is a packing 5-coloring of Θ. To see that, we will show that for any two distinct vertices x and y with $\varphi(x) = \varphi(y) = c$, $c \in \{1, 2, 3\}$, we have $d_\Theta(x, y) > c$ (the case $c \in \{4, 5\}$ does not need to be considered since there is only one vertex with color 4 and one vertex with color 5). Note first that the restriction of φ to any path P_i is a packing coloring of P_i. Hence, we just need to consider the case when x and y do not belong to the same path. If $c = 1$, the property obviously holds since only internal vertices are colored with color 1. Since at most one vertex with color 2 is adjacent to u and at most one vertex with color 2 is adjacent to v, the property also holds when $c = 2$. Since at most one vertex with color 3 is adjacent to u and no vertex with color 3 is adjacent to v, the property also holds when $c = 3$. Hence, φ is a packing 5-coloring of Θ.

Finally, when $n_3 > 3$, we color three paths of length 3 as above and the remaining ones using distinct patterns of the form 4165, 4175, etc. Since each color $c > 5$ is used only once, we clearly get a packing $(n_3 + 2)$-coloring of Θ.

The fact that $\max\{5, n_3 + 2\}$ is a tight upper bound whenever $n_3 \geq 3$ follows from Lemma 8 proven below. ■

We will now characterize generalized theta graphs with packing chromatic number k for every $k \geq 3$. Since every cycle is a generalized theta graph, we know by Theorem 2(2) that $\chi_\rho(\Theta) \geq 3$ for every generalized theta graph Θ. Moreover, Theorem 2(2) characterizes generalized theta graphs $\Theta_{\ell_1,\dots,\ell_p}$ with packing chromatic number 3 and 4 whenever $p = 2$. Therefore, unless otherwise specified, we will always consider $p \geq 3$ in the rest of this section.

The next lemma determines the packing chromatic number of generalized theta graphs of the form $\Theta_{3,\dots,3}$:

Lemma 8 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, with $n_3 = p$. We then have $\chi_\rho(\Theta) = p + 2$.*

Proof. By Theorem 7, we have $\chi_\rho(\Theta) \leq p + 2$. Therefore, it is enough to prove that for every packing k-coloring π of Θ, $k \geq p + 2$.

If $\pi(u) = \pi(v) = 1$ then at most two remaining vertices can be assigned color 2 and all other remaining vertices must be assigned distinct colors, so that π uses at least $2(p - 1) + 2 = 2p \geq p + 2$ colors.

If $\pi(u) = 1$ and $\pi(v) \neq 1$, then none of the vertices x_i^1, $1 \leq i \leq p$, can be assigned color 1 and, since the $p+1$ vertices $\{v, x_1^1, \dots, x_p^1\}$ are pairwise at distance 2, they must be assigned distinct colors so that π must use at least $p+2$ colors. The case $\pi(v) = 1$ and $\pi(u) \neq 1$ is similar.

Finally, if $\pi(u) \neq 1$ and $\pi(v) \neq 1$, then at most p internal vertices can be assigned color 1 (one per path). Since any two internal vertices are at distance at most 3 from each other and from u and v, and no three such vertices (including u and v) are pairwise at distance 3 from each other, color 2 can be used at most twice, so that π must use at least $p+2$ colors. ■

The following lemma characterizes generalized theta graphs with packing chromatic number k for every $k > 5$:

Lemma 9 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, be a generalized theta graph. Then, for every $k > 5$, $\chi_\rho(\Theta) = k$ if and only if $n_3 = k - 2$.*

Proof. If $n_3 = k - 2$, we get $\chi_\rho(\Theta) \leq k$ by Theorem 7, and $\chi_\rho(\Theta) \geq k$ by Lemma 8 and Proposition 1.

If $\chi_\rho(\Theta) = k$, we get $n_3 \geq k - 2$ by Theorem 7 and $n_3 \leq k - 2$ by Lemma 8. ■

Generalized theta graphs with packing chromatic number 3 are characterized as follows.

Lemma 10 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 2$, be a generalized theta graph. We then have $\chi_\rho(\Theta) = 3$ if and only if one of the following conditions holds:*

1. *$\ell_1 = 1$ and $\ell_2 = \dots = \ell_p = 2$, or*
2. *for every i and j, $1 \leq i < j \leq p$, $\ell_i + \ell_j \equiv 0 \pmod 4$.*

Proof. By Theorem 2(2), if $p = 2$ then $\chi_\rho(\Theta) = 3$ if and only if $\ell_1 = 1$ and $\ell_2 = 2$, or $\ell_1 + \ell_2 \equiv 0 \pmod 4$. Therefore, assume $p \geq 3$.

We first prove that if $\ell_1 = 1$, $\ell_2 = 2$ and $\ell_p > 2$ then $\chi_\rho(\Theta) > 3$. Assume to the contrary that there exists a packing 3-coloring π of Θ. Since P_1 and P_2 induce a cycle of length 3, we necessarily have $\pi(x_2^1) = \pi(x_p^1) = 1$ and, without loss of generality, $\pi(u) = 2$ and $\pi(v) = 3$, which implies that no color is available for x_p^2 since $d_\Theta(x_p^2, x_p^1) = 1$, $d_\Theta(x_p^2, u) = 2$ and $d_\Theta(x_p^2, v) \leq 3$, a contradiction.

We know by Theorem 2(2) that, for every $n \geq 3$, $3 \leq \chi_\rho(C_n) \leq 4$ and $\chi_\rho(C_n) = 3$ if and only if $n = 3$ or $n \equiv 0 \pmod 4$. Therefore, if Θ contains a cycle of length $\ell \not\equiv 0 \pmod 4$,

$\ell > 3$, then $\chi_\rho(\Theta) > 3$. Clearly, this happens whenever there exist i and j, $1 \leq i < j \leq p$, with $\ell_i + \ell_j = \ell$.

Conversely, assume that for every i and j, $1 \leq i < j \leq p$, $\ell_i + \ell_j \equiv 0 \pmod 4$. We have two cases to consider. If $\ell_i \equiv 0 \pmod 4$ for every i, $1 \leq i \leq p$, a packing 3-coloring π of Θ is obtained by coloring each path P_i with the color pattern (2131)*2. If $\ell_i \equiv 2 \pmod 4$ for every i, $1 \leq i \leq p$, a packing 3-coloring π of Θ is obtained by coloring each path P_i with the color pattern 21(3121)*3.

This completes the proof. ■

It remains to characterize generalized theta graphs with packing chromatic number 4 and 5. Thanks to Theorem 2(2), we do not need to consider cycles. The following series of lemmas will allow us to characterize generalized theta graphs (assuming $p \geq 3$) with packing chromatic number at most 4, depending on the colors assigned to the end-vertices u and v.

The first three lemmas characterize generalized theta graphs that admit a packing 4-coloring π with $\pi(u) = \pi(v) = 4, 3$ or 2.

Lemma 11 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, be a generalized theta graph. There exists a packing 4-coloring π of Θ with $\pi(u) = \pi(v) = 4$ if and only if $n_1 = n_2 = n_3 = n_4 = 0$.*

Proof. Suppose first that π is a packing 4-coloring of Θ with $\pi(u) = \pi(v) = 4$. We then necessarily have $d(u, v) > 4$, which implies $n_1 = n_2 = n_3 = n_4 = 0$.

Conversely, suppose that $n_1 = n_2 = n_3 = n_4 = 0$. We can color each path P_i, $1 \leq i \leq p$, of length $\ell_i \geq 5$, using the following patterns, depending on the value of ($\ell_i \mod 4$):

- $4(1213)^+1214$, if $\ell_i \equiv 0 \pmod 4$,
- $413(1213)^*214$, if $\ell_i \equiv 1 \pmod 4$,
- $4(1213)^+14$, if $\ell_i \equiv 2 \pmod 4$,
- $4(1213)^+214$, if $\ell_i \equiv 3 \pmod 4$.

The so-obtained 4-coloring is clearly a packing 4-coloring of Θ. ■

Lemma 12 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, be a generalized theta graph. There exists a packing 4-coloring π of Θ with $\pi(u) = \pi(v) = 3$ if and only if $n_1 = n_2 = n_3 = 0$, $n_5 \leq 2$ and $n_5 + n_6 \leq 4$.*

Proof. Suppose first that π is a packing 4-coloring of Θ with $\pi(u) = \pi(v) = 3$. We then necessarily have $d(u, v) > 3$, which implies $n_1 = n_2 = n_3 = 0$. Note that we can only use colors 1, 2 and 4 for coloring the internal vertices of each path P_i with $\ell_i \leq 7$, $1 \leq i \leq p$. Therefore, each coloring of a path of length 5 must use once the color 4, which implies $n_5 \leq 2$, since otherwise we would have two vertices with color 4 at distance at most 4 from each other. Similarly, a path of length 6 can only be colored 3121413, 3141213, 3121423, 3241213, 3124123, 3214213 or 3214123, which implies $n_6 \leq 4$ (again, due to vertices with colour 4). Moreover, we necessarily have $n_6 \leq 2$ whenever $n_5 = 2$ and $n_6 \leq 3$ whenever $n_5 = 1$, which gives $n_5 \leq 2$ and $n_5 + n_6 \leq 4$.

Conversely, suppose that $n_1 = n_2 = n_3 = 0$, $n_5 \leq 2$ and $n_5 + n_6 \leq 4$. We color each path of length 4 with 31213. If $n_5 = 2$, we color the two paths of length 5 with 312413 and 314213 and the (at most two) paths of length 6 with 3124123 and 3214213. If $n_5 = 1$, we color the path of length 5 with 312413 and the (at most three) paths of length 6 with 3141213, 3124123 and 3214213. If $n_5 = 0$, we color the (at most four) paths of length 6 with 3121413, 3141213, 3124123 and 3214213.

Finally, we color each path P_i, $1 \leq i \leq p$, of length $\ell_i \geq 7$, using the following patterns, depending on the value of ($\ell_i \mod 4$):

- $3(1213)^+1213$, if $\ell_i \equiv 0 \pmod 4$,
- $3(1213)^+41213$, if $\ell_i \equiv 1 \pmod 4$,
- $3(1213)^+141213$, if $\ell_i \equiv 2 \pmod 4$,
- $3(1213)^*1241213$, if $\ell_i \equiv 3 \pmod 4$.

The so-obtained 4-coloring is clearly a packing 4-coloring of Θ. ■

Lemma 13 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, be a generalized theta graph. There exists a packing 4-coloring π of Θ with $\pi(u) = \pi(v) = 2$ if and only if $n_1 = n_2 = 0$ and one of the following conditions holds:*

1. *$n_3 = 1$ and $n_5 + n_6 + n_7 = 0$, or*
2. *$n_3 = 0$ and $n_5 + n_6 + n_7 \leq 2$.*

Proof. Suppose first that π is a packing 4-coloring of Θ with $\pi(u) = \pi(v) = 2$. We then necessarily have $d(u, v) > 2$, which implies $n_1 = n_2 = 0$. Note that we can only use colors 1, 3 and 4 for coloring the internal vertices of each path P_i with $\ell_i \leq 5$, $1 \leq i \leq p$. Therefore, a path of length 3 can only be colored either 2132 (or 2312) or 2142 (or 2412), which implies $n_3 \leq 2$.

If $n_3 = 2$ then, without loss of generality, the two paths of length 3 are colored either 2132 and 2142, or 2132 and 2412. In both cases, no path of length $\ell \geq 4$ can be colored since only the color 1 is available for the vertices at distance 1 and 2 from v. This implies $p = 2$, contradicting the assumption $p \geq 3$.

If $n_3 = 1$, as observed above, the corresponding path of length 3 is either colored 2132 (or 2312) or 2142 (or 2412). In the former case (assume, without loss of generality, that the path is colored 2132), every other vertex at distance at most 2 from v must be assigned colored 1 or 4, which implies $\sum_{\ell\geq 4} n_\ell \leq 1$, so that $p = 2$, contrary to the assumption $p \geq 3$. The corresponding path of length 3 is thus colored 2142 (or 2412), so that every other vertex at distance at most 2 from u or v must then be colored 1 or 3. There is no such coloring for a path of length 5, only one such coloring for a path of length 6, namely 2314132, and two such colorings for a path of length 7, up to symmetry, namely 23124132 and 21324132. Since each of these colorings uses color 3 on a neighbor of u or v we necessarily have $n_5 + n_6 + n_7 \leq 1$. If $n_5 + n_6 + n_7 = 1$ then, again, no other path can be colored since only the color 1 is available for the vertices at distance 1 and 2 from v (or u), which implies $p = 2$, contradicting the assumption $p \geq 3$. Therefore, $n_5 + n_6 + n_7 = 0$ and condition (i) is satisfied.

If $n_3 = 0$ then, as observed above, every path of length 5, 6 or 7 must contain a vertex with color 4 at distance at most 2 from u or v since $p \geq 3$. Therefore, at most two such paths can occur, that is $n_5 + n_6 + n_7 \leq 2$, and thus condition (ii) is satisfied.

Conversely, suppose that $n_1 = n_2 = 0$. If $n_3 = 1$ and $n_5 + n_6 + n_7 = 0$, we color the path of length 3 with 2142 and every path of length 4 with 21312. We then color each path P_i, $1 \leq i \leq p$, of length $\ell_i \geq 8$, using the following patterns, depending on the value of (ℓ_i mod 4):

- $2(1312)^+1312$, if $\ell_i \equiv 0 \pmod 4$,
- $2(1312)^+41312$, if $\ell_i \equiv 1 \pmod 4$,
- $2(1312)^+121312$, if $\ell_i \equiv 2 \pmod 4$,
- $2(1312)^+4121312$, if $\ell_i \equiv 3 \pmod 4$.

If $n_3 = 0$ and $n_5 + n_6 + n_7 \leq 2$ we first color each path P_i, $1 \leq i \leq p$, of length ℓ_i, $4 \leq \ell_i \leq 7$, as follows:

- 21312, if $\ell_i = 4$,
- 213412 or 214312, if $\ell_i = 5$,
- 2131412 or 2141321, if $\ell_i = 6$,
- 21321412 or 21412312, if $\ell_i = 7$.

Note that if $n_5+n_6+n_7=2$ the two corresponding paths must use the patterns $214\dots2$ and $2\dots412$ so that the distance between the two vertices with color 4 is at least 5. We then color each path P_i, $1\le i\le p$, of length $\ell_i\ge 8$, depending on the value of $(\ell_i \mod 4)$ as in the previous case.

The so-obtained 4-coloring is clearly a packing 4-coloring of Θ. ■

The next three lemmas characterize generalized theta graphs that admit a packing 4-coloring π with $\pi(u),\pi(v)\in\{2,3,4\}$, $\pi(u)\neq\pi(v)$.

Lemma 14 *Let $\Theta=\Theta_{\ell_1,\dots,\ell_p}$, $p\ge 3$, be a generalized theta graph. There exists a packing 4-coloring π of Θ with $\pi(u)=3$ and $\pi(v)=4$ if and only if one of the following conditions holds:*

1. *$n_1\le 1$, $n_3\le 2$ and $n_5=n_6=0$, or*
2. *$n_1=0$, $n_3\le 2$ and ($n_5=0$ or $n_5+n_6\le 1$).*

Proof. Suppose first that π is a packing 4-coloring of Θ with $\pi(u)=3$ and $\pi(v)=4$.

There are only two possible colorings of a path of length 3, namely 3124 and 3214, which implies that we can have at most two such paths (otherwise, we would have two vertices with color 2 at distance 2 from each other).

Suppose first that $n_1=1$. In that case, since every internal vertex of a path of length 5 or 6 is at distance at most 3 from u and v, the only available colors for these vertices are 1 and 2, so that $n_5+n_6=0$ and condition (i) is satisfied.

Suppose now that $n_1=0$. Since the only possible coloring of a path of length 5 is 312134, we necessarily have $n_5\le 1$. Consider the possible colorings of a path of length 6. Since color 4 can only be used on the neighbor of u, the four other internal vertices must use color 3 and thus color 3 has to be used on a vertex at distance at most 2 from v. This implies $n_6=0$ whenever $n_5=1$ and thus condition (ii) is satisfied.

We finally prove that for every generalized theta graph satisfying any of these conditions, there exists a packing 4-coloring π with $\pi(u)=3$ and $\pi(v)=4$. Every path of length 2 can be colored 314 and every path of length 4 can be colored 31214. If $n_3=1$ the path of length 3 can be colored 3124, and if $n_3=2$ the paths of length 3 can be colored 3124 and 3214. If $n_1=0$ and $n_5=1$, the path of length 5 can be colored 312134. If $n_1=0$ and $n_5=0$, all the paths of length 6 can be colored 3121314.

It remains to prove that every path P_i, $1\le i\le p$, of length $\ell_i\ge 7$ can be colored. This can be done using the following patterns, depending on the value of $(\ell_i \mod 4)$:

- $3(1213)^{+}1214$ if $\ell_i \equiv 0 \pmod 4$,
- $31214312(1312)^{*}14$ if $\ell_i \equiv 1 \pmod 4$,
- $31214(1312)^{+}14$ if $\ell_i \equiv 2 \pmod 4$,
- $3(1213)^{+}214$ if $\ell_i \equiv 3 \pmod 4$.

The so-obtained 4-coloring is clearly a packing 4-coloring of Θ. ■

Lemma 15 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, be a generalized theta graph. There exists a packing 4-coloring π of Θ with $\pi(u) = 2$ and $\pi(v) = 4$ if and only if one of the following conditions holds:*

1. $n_1 \leq 1$ *and* $n_3 = n_7 = 0$, *or*
2. $n_1 \leq 1$ $n_3 = 0$, $n_7 \leq 1$ *and* $n_8 = 0$, *or*
3. $n_1 = n_3 \leq 1$ *and* $n_4 = n_7 = n_8 = 0$, *or*
4. $n_1 = n_2 = n_3 = 0$ *and* $n_7 \leq 1$, *or*
5. $n_1 = n_2 = n_3 = n_4 = 0$, $n_7 = 2$ *and* $n_8 = 0$, *or*
6. $n_1 = n_2 = 0$, $n_3 \leq 1$, $n_4 = 0$ *and* $n_7 + n_8 \leq 1$, *or*
7. $n_1 = n_3 = n_7 = 0$, *or*
8. $n_1 = n_3 = n_4 = 0$, $n_7 \leq 1$ *and* $n_8 = 0$, *or*
9. $n_1 = 0$, $n_3 \leq 1$ *and* $n_4 = n_7 = n_8 = 0$.

Proof. Suppose first that π is a packing 4-coloring of Θ with $\pi(u) = 2$ and $\pi(v) = 4$.

Since every path of length 3 can be colored either 2134 or 2314, we necessarily have $n_3 \leq 1$ (otherwise, we would have two vertices with color 3 at distance 3 from each other). Moreover, since every path of length 4 can be colored either 21314 or 21324, therefore using color 3 at distance 2 from u and v, we necessarily have $n_4 = 0$ whenever a path uses color 3 on a neighbor of u or v (thus, in particular if $n_3 = 1$).

Suppose that $n_1 = 1$ and consider the possible colorings of a path of length 7. On its internal vertices, color 4 cannot be used, color 3 can be used only twice, color 2 can be used only once and color 1 can be used three times.

Therefore, the only possible colorings of a path of length 7 are 21312134 and 23121314. We thus necessarily have $n_7 \leq 1$, and $n_7 = 0$ whenever $n_3 = 1$, otherwise we would have two vertices with color 3 at distance 2 or 3 from each other. Similarly, for the internal vertices of a path of length 8, color 4 cannot be used, color 3 and 2 can both be used at most twice, and color 1 can be used at most four times. Therefore, the only colorings of a path of length 8 are 213121314 and 231213214. We thus necessarily have $n_8 = 0$ whenever $n_3 = 1$ or $n_7 = 1$, again because of vertices with color 3.

Therefore, one of the conditions (i), (ii) or (iii) is satisfied.

Suppose now that $n_1 = 0$. We already know that $n_3 \leq 1$, and that $n_4 = 0$ whenever $n_3 = 1$.

Now, the possible colorings of a path of length 7 are 21312134 and 21431214 (using the color 3 or 4 on the neighbour of u cannot give a better coloring than these two colorings). This implies $n_7 \leq 2$ (because of vertices with color 3 or 4) and both these colorings must be used when $n_7 = 2$. Moreover, if $n_3 = 1$ then the coloring 21312134 cannot be used and thus $n_7 \leq 1$ in that case. On the other hand, the coloring 21431214 cannot be used whenever $n_2 \geq 1$. Similarly, the possible colorings of a path of length 8 are 213121314 and 214131214 (again, using the color 3 or 4 on the neighbour of u cannot give a better coloring than these two colorings). If $n_2 \geq 1$, the coloring 214131214 cannot be used. On the other hand, the coloring 213121314 cannot be used whenever $n_3 = 1$, or $n_2 \geq 1$ and $n_7 = 1$, or $n_2 = 0$ and $n_7 = 2$, because of vertices with color 3.

Therefore, one of the conditions (iv) to (ix) must hold.

We finally prove that for every generalized theta graph satisfying any of these conditions, there exists a packing 4-coloring π with $\pi(u) = 2$ and $\pi(v) = 4$. We first color all the paths P_i, $1 \leq i \leq p$, of length $\ell_i \notin \{3, 7, 8\}$, if any, as follows:

- $\ell_i = 2$: 214,
- $\ell_i = 4$: 21314,
- $\ell_i = 5$: 213214,
- $\ell_i = 6$: 2131214,
- $\ell_i \geq 9$: for these paths, we use the following patterns, depending on the value of (ℓ_i mod 4):
  - $2(1312)^+14131214$ if $\ell_i \equiv 0 \pmod 4$,
  - $2(1312)^+13214$ if $\ell_i \equiv 1 \pmod 4$,
  - $2(1312)^+14$ if $\ell_i \equiv 2 \pmod 4$,
  - $2(1312)^+1431214$ if $\ell_i \equiv 3 \pmod 4$.

It remains to colors the paths of length 3, 7 or 8. This can be done according to the condition of the Lemma that is satisfied:

1. All the paths are already colored.
2. The path of length 7 is colored 21312134 (recall that we have no path of length 3 in that case).
3. The path of length 3 is colored 2134 (recall that we have no path of length 4 in that case).
4. The path of length 7, if any is colored 21431214 and all the paths of length 8 are colored 213121314 (recall that we have no path of length 1, 2 or 3 in that case).
5. The two paths of length 7 are colored 21312134 and 21431214 (recall that we have no path of length less than 5 in that case).
6. The path of length 3 is colored 2134, the path of length 7, if any, is colored 21431214 and the path of length 8, if any, is colored 214131214 (recall that we have no path of length 1, 2 or 4 and at most one path of length either 7 or 8 in that case).
7. All the paths are already colored.
8. The path of length 7 is colored 21312134 (recall that we have no path of length 1, 3, 4 or 8 in that case).
9. The path of length 3 is colored 2134 (recall that we have no path of length 1, 4, 7 or 8 in that case).

In all cases, the so-obtained 4-coloring is clearly a packing 4-coloring of Θ. ■

Lemma 16 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, be a generalized theta graph. There exists a packing 4-coloring π of Θ with $\pi(u) = 2$ and $\pi(v) = 3$ if and only if one of the following conditions holds:*

1. $n_1 \leq 1$, and $\sum_{i\geq 3} n_i \leq 1$, or

2. $n_1 = 0$ and $n_3 + n_4 + n_5 \leq 1$.

Proof. Suppose first that π is a packing 4-coloring of Θ with $\pi(u) = 2$ and $\pi(v) = 3$.

If $n_1 = 1$ then all the neighbors of u and v must be colored 1 or 4. In every path of length $\ell \geq 3$, the vertex at distance 2 from u is then necessarily colored 4 if the neighbor of u is colored 1, or 1 if the neighbor of u is colored 4. Hence, we can have at most one such path (otherwise, we would have two vertices with color 4 at distance at most 4 from each other), that is $\sum_{i\geq 3} n_i \leq 1$ and condition (i) is satisfied.

If $n_1 = 0$ then, since every path of length 3 or 4 must use the color 4 on a vertex at distance at most 2 from u and v, and every path of length 5 must use the color 4 on a vertex at distance at most 2 from v, we necessarily have $n_3 + n_4 + n_5 \leq 1$ and condition (ii) is satisfied.

We finally prove that for every generalized theta graph satisfying any of these conditions, there exists a packing 4-coloring π with $\pi(u) = 2$ and $\pi(v) = 3$. Every path of length 2 can be colored 213. If there is a path of length 3 (which implies either $n_1 = 1$ and $\sum_{i\geq 4} n_i = 0$, or $n_1 = n_4 = n_5 = 0$), then we color this path with 2143. If there is a path of length 4 (which implies either $n_1 = 1$, $n_3 = 0$ and $\sum_{i\geq 5} n_i = 0$, or $n_1 = n_3 = n_5 = 0$), then we color this path with 21413. If there is a path of length 5 (which implies either $n_1 = 1$, $n_3 = n_4 = 0$ and $\sum_{i\geq 6} n_i = 0$, or $n_1 = n_3 = n_4 = 0$), then we color this path with 214213.

It remains to prove that every path P_i of length $\ell_i \geq 6$ can be colored. If $n_1 = 1$ then $n_3 = n_4 = n_5 = 0$ and we have only one such path. We then color this path using one of the following patterns, depending on the value of $(\ell_i \mod 4)$:

- $214(1312)^+13$ if $\ell_i \equiv 0 \pmod 4$,
- $2142(1312)^+13$ if $\ell_i \equiv 1 \pmod 4$,
- $21412(1312)^*13$ if $\ell_i \equiv 2 \pmod 4$,
- $214312(1312)^*13$ if $\ell_i \equiv 3 \pmod 4$.

If $n_1 = 0$, we color any such path using the following patterns, depending on the value of $(\ell_i \mod 4)$:

- $2(1312)^*14131213$ if $\ell_i \equiv 0 \pmod 4$,
- $2(1312)^+14213$ if $\ell_i \equiv 1 \pmod 4$,
- $2(1312)^+13$ if $\ell_i \equiv 2 \pmod 4$,
- $2(1312)^*1341213$ if $\ell_i \equiv 3 \pmod 4$.

The so-obtained 4-coloring is clearly a packing 4-coloring of Θ. ■

Using Lemmas 11 to 16 we get a complete characterization of generalized theta graphs admitting a packing 4-coloring that does not use color 1 on vertex u nor on vertex v:

Theorem 17 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, be a generalized theta graph. There exists a packing 4-coloring π of Θ with $\pi(u) \neq 1$ and $\pi(v) \neq 1$ if and only if one of the following conditions holds:*

1. $n_1 = n_2 = n_3 = n_4 = 0$,
2. $n_1 = n_2 = n_3 = 0$, $n_5 \leq 2$ *and* $n_5 + n_6 \leq 4$,
3. $n_1 = n_2 = n_3 = 0$ *and* $n_7 \leq 1$,
4. $n_1 = n_2 = n_3 = 0$ *and* $n_5 + n_6 + n_7 \leq 2$,
5. $n_1 = n_2 = 0$, $n_3 \leq 1$ *and* $n_5 = n_6 = n_7 = 0$,
6. $n_1 = n_2 = 0$, $n_3 \leq 1$, $n_4 = 0$ *and* $n_7 + n_8 \leq 1$,
7. $n_1 = n_3 = n_4 = 0$, $n_7 \leq 1$ *and* $n_8 = 0$,
8. $n_1 = 0$, $n_3 \leq 2$ *and* ($n_5 = 0$ *or* $n_5 + n_6 \leq 1$),
9. $n_1 = 0$ *and* $n_3 + n_4 + n_5 \leq 1$,
10. $n_1 \leq 1$, $n_3 \leq 2$ *and* $n_5 = n_6 = 0$,
11. $n_1 \leq 1$, *and* $n_3 = n_7 = 0$,
12. $n_1 \leq 1$, $n_3 = 0$, $n_7 \leq 1$ *and* $n_8 = 0$,
13. $n_1 \leq 1$, $n_3 \leq 1$ *and* $n_4 = n_7 = n_8 = 0$,
14. $n_1 \leq 1$, *and* $\sum_{i\geq 3} n_i \leq 1$.

Proof. This theorem simply summarizes the results of Lemmas 11 to 16:

- Item 1 follows from Lemma 11 and contains case (v) of Lemma 15.
- Item 2 follows from Lemma 12.
- Item 3 follows from case (iv) of Lemma 15.
- Item 4 follows from case (ii) of Lemma 13.
- Item 5 follows from case (i) of Lemma 13.
- Item 6 follows from case (vi) of Lemma 15.
- Item 7 follows from case (viii) of Lemma 15.
- Item 8 follows from case (ii) of Lemma 14.
- Item 9 follows from case (ii) of Lemma 16.
- Item 10 follows from case (i) of Lemma 14.
- Item 11 follows from case (i) of Lemma 15 and contains (vii) of Lemma 15.

- Item 12 follows from case (ii) of Lemma 15.
- Item 13 follows from case (iii) Lemma 15 and contains case (ix) of Lemma 15.
- Item 14 follows from case (i) of Lemma 16.

Hence, all the cases have been considered, this concludes the proof. ■

If a generalized theta graph Θ satisfies none of the conditions 1 to 14 of Theorem 17, then every packing 4-coloring of Θ must use color 1 on u or v. The following observation will be useful:

Observation 18 *If a generalized theta graph $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 3$, admits a packing 4-coloring π with $\pi(u) = 1$ then we necessarily have $p = 3$.*

To see that, it suffices to note that no two neighbors of u can be assigned the same color and that none of them can be colored 1.

The next lemma will show that, with one exception, no generalized theta graph satisfying none of the conditions 1 to 14 admits a packing 4-coloring. By Observation 18, it suffices to consider generalized theta graphs of the form $\Theta_{\ell_1,\ell_2,\ell_3}$. Moreover, by symmetry, it suffices to consider packing 4-colorings that assign the color 1 to u.

Lemma 19 *If $\Theta = \Theta_{\ell_1,\ell_2,\ell_3}$ is a generalized theta graph and π a packing 4-coloring of Θ with $\pi(u) = 1$, then Θ satisfies at least one of the conditions 1 to 14, except if $\Theta = \Theta_{1,7,8}$.*

Proof. We consider two cases, according to the value of n_1.

1. $n_1 = 0$.
   If $n_2 = n_3 = n_4 = 0$ then Θ satisfies condition 1.

   Observe that we cannot have $n_3 = 3$ since, by Lemma 8, we would have $\chi_\rho(\Theta) = 5$, a contradiction.

   If $n_3 = 2$ then we necessarily have $n_5 + n_6 \leq 1$ and therefore Θ satisfies condition 8.

   If $n_3 = 1$ and $n_5 = 0$ then Θ satisfies condition 8. If $n_3 = 1$, $n_5 = 1$ and $n_6 = 0$ then, again, Θ satisfies condition 8. If $n_3 = 1$, $n_5 = 1$ and $n_6 = 1$ then, since we necessarily have $n_2 = n_4 = n_7 = n_8 = 0$, Θ satisfies condition 6. If $n_3 = 1$ and $n_5 = 2$ then Θ also satisfies condition 6.

   Suppose that $n_3 = 0$. If $n_4 \geq 1$ and $n_2 = 0$ then we necessarily have $n_5 \leq 2$ and $n_5 + n_6 \leq 4$ and Θ satisfies condition 2. If $n_4 \geq 1$ and $n_2 \geq 1$ then we necessarily have $n_5 + n_6 \leq 1$ and Θ satisfies condition 8. If $n_4 = 0$ and $n_5 \leq 1$ then Θ satisfies condition 9. If $n_4 = 0$ and $n_5 \geq 2$ then Θ satisfies condition 6 if $n_7 = 0$ or condition 7 if $n_7 = 1$ (since we then have $n_7 + n_8 \leq 1$).

2. $n_1 = 1$.
   In that case, we necessarily have $n_3 \leq 2$. If $n_3 = 2$ then we necessarily have $n_5 = n_6 = 0$ and Θ satisfies condition 10.

   If $n_3 = 1$ and $n_5 = n_6 = 0$ then Θ satisfies condition 10. If $n_3 = 1$ and $n_5 + n_6 = 1$ then we necessarily have $n_4 = n_7 = n_8 = 0$ and Θ satisfies condition 13.

   Suppose that $n_3 = 0$. If $n_7 = 0$ then Θ satisfies condition 11. If $n_7 = 1$ and $n_8 = 0$ then Θ satisfies condition 12. If $n_7 = 2$ then we necessarily have $n_5 = n_6 = 0$ and Θ satisfies condition 10.

   There is now only one remaining case, namely $n_1 = n_7 = n_8 = 1$. In that case, the three paths of the generalized theta graph $\Theta_{1,7,8}$ can be colored 12, 14121312 and 132141312, respectively.

This completes the proof. ■

We are now able to characterize generalized theta graphs with packing chromatic number 4:

Theorem 20 *Let $\Theta = \Theta_{\ell_1,\dots,\ell_p}$, $p \geq 2$, be a generalized theta graph. We then have $\chi_\rho(\Theta) = 4$ if and only if either*

1. *$p = 2$, $\ell_1 + \ell_2 \neq 3$ and $\ell_1 + \ell_2 \not\equiv 0 \pmod 4$, or*
2. *$p \geq 3$, $n_2 \neq p$, there exist i_1, i_2, $1 \leq i_1 < i_2 \leq p$, such that $\ell_{i_1} + \ell_{i_2} \not\equiv 0 \pmod 4$, and Θ satisfies one of the conditions 1 to 14,*
3. *$\Theta = \Theta_{1,7,8}$.*

Proof. If $p = 2$ the result follows from Theorem 2(2). If $p \geq 3$, the result follows from Theorem 17 (case 2) or from Lemma 19 (case 3). ■

Using Lemma 9, Lemma 10 and Theorem 20, we get that the packing chromatic number of any generalized theta graph $\Theta = \Theta_{\ell_1,\dots,\ell_p}$ can be computed in time $O(p)$.

3 Oriented generalized theta graphs

In this section, we study the packing chromatic number of oriented generalized theta graphs $\Theta_{\ell_1,\dots,\ell_p}$. Recall that we denote by n_ℓ, $\ell \geq 1$, the number of paths of length ℓ, that is

$$n_\ell = |\{i \ / \ 1 \leq i \leq p,\ \ell_i = \ell\}|.$$

We prove the following:

$4 \longrightarrow 1 \longrightarrow 2 \longrightarrow 5$	$41 \longrightarrow 2 \longrightarrow 5$
$4 \longrightarrow 3 \longrightarrow 15$	$42 \longrightarrow 15$
$4 \longrightarrow 12 \longrightarrow 5$	$412 \longrightarrow 5$
$4 \longrightarrow 315$	4215

Figure 1: Coloring of oriented paths of length 3 (proof of Theorem 21)

Theorem 21 *For every oriented generalized theta graph* $\Theta = \Theta_{\ell_1,\dots,\ell_p}$*,* $p \geq 2$*,* $2 \leq \chi_\rho(\Theta) \leq 5$ *and these two bounds are tight.*

Proof. It follows from Proposition 4 that 2 is a tight lower bound for $\chi_\rho(\Theta)$. By Proposition 3 and Theorem 7, we know that $\chi_\rho(\Theta) \leq 5$ whenever $n_3 \leq 3$.

Assume thus that $n_3 > 3$. Let us denote by P_i the orientation of the path P_i for every i, $1 \leq i \leq p$, and let $\varphi : V(\Theta) \longrightarrow \{1, \dots, 5\}$ be the mapping defined as in the proof of Theorem 7, except for the internal vertices of the paths P_i with $\ell_i = 3$, which are colored as shown in Figure 1, according to their orientation.

We claim that φ is a packing 5-coloring of Θ. Again, the restriction of φ to any path P_i is a packing coloring of P_i. Moreover, from the proof of Theorem 7, we know that the restriction of φ to $\bigcup\{P_i :\ \ell_i \neq 3\}$ is a packing 5-coloring. Hence, we just need to prove that for any two distinct vertices x and y with $x \in P_i$, $\ell_i = 3$, $\varphi(x) = \varphi(y) = c$, $c \in \{2, 3\}$ and $\{x, y\} \cap \{u, v\} = \emptyset$, we have $d_\Theta(x, y) > c$.

Suppose first that $c = 2$. Since every vertex y in P_j, $\ell_j \neq 3$, with $\varphi(y) = 2$ is at weak directed distance at least 2 from u and v, no conflict can occur between x and y. If y belongs to some P_j with $\ell_j = 3$ then the possible arcs are only xu, yu, xv and yv (see Figure 1), and no conflict can occur between x and y.

Suppose now that $c = 3$. In that case, $x = x_i^1$ and ux is an arc (see Figure 1). Since every vertex y in P_j, $\ell_j \neq 3$, with $\varphi(y) = 3$ is at weak directed distance at least 3 from u and at least 2 from v, no conflict can occur between x and such a y. If y belongs to some P_j with $\ell_j = 3$ then $y = y_i^1$ and uy is an arc, so that there is no conflict between x and y.

We thus get $\chi_\rho(\Theta) \leq 5$.

Let us now prove that this bound is tight. For that, consider the oriented generalized theta graph Θ_0 obtained by identifying (according to their name, either u or v) the end-vertices of the six following directed paths:

$$ux_1x_2x_3x_4v,\quad uy_1y_2v,\quad uz_1v,\quad vx'_1x'_2x'_3x'_4u,\quad vy'_1y'_2u,\quad vz'_1u.$$

We claim that $\chi_\rho(\Theta_0) = 5$. To see that, suppose that there exists a packing 4-coloring π of Θ_0. We consider five cases, according to the values of $\pi(u)$ and $\pi(v)$ (up to symmetry). Note that since $d_{\Theta_0}(u, v) = 2$, we necessarily have $\pi(u) = \pi(v) = 1$ whenever $\pi(u) = \pi(v)$.

1. $\pi(u) = \pi(v) = 1$.
   In that case, no vertex in $\{y_1, y_2, y'_1, y'_2, z_1, z'_1\}$ can be colored 1. Moreover, since any two vertices in $\{y_1, y_2, y'_1, y'_2\}$ are linked by a directed path (in either direction) of length at most 3, we necessarily have either $\pi(y_1) = \pi(y'_1) = 2$ and $\{\pi(y_2), \pi(y'_2)\} = \{3, 4\}$ or $\pi(y_2) = \pi(y'_2) = 2$ and $\{\pi(y_1), \pi(y'_1)\} = \{3, 4\}$. In both cases, one vertex in $\{z_1, z'_1\}$ cannot be colored.

2. $\pi(u) = 1$, $\pi(v) \in \{2, 3\}$.
   If $\pi(v) = 2$ (resp. $\pi(v) = 3$), we necessarily have $\{\pi(z_1), \pi(z'_1)\} = \{3, 4\}$ (resp. $\{\pi(z_1), \pi(z'_1)\} = \{2, 4\}$). If $\pi(z_1) = 4$ (resp. $\pi(z'_1) = 4$), then $\{\pi(y'_1), \pi(y'_2)\} = \{3\}$ (resp. $\{\pi(y_1), \pi(y_2)\} = \{3\}$), a contradiction.

3. $\pi(u) = 1$, $\pi(v) = 4$.
   In that case, we necessarily have $\{\pi(z_1), \pi(z'_1)\} = \{2, 3\}$, which implies $\{\pi(y_1), \pi(y'_2)\} = \{2, 3\}$ and $\pi(y_2) = \pi(y'_1) = 1$. If $\pi(z_1) = 2$ then $\pi(x_1) = 2$ and $\pi(x_2) = 1$, so that x_3 cannot be colored. If $\pi(z'_1) = 2$ then $\pi(x'_1) = 2$ and $\pi(x'_2) = 1$, so that x'_3 cannot be colored.

4. $\pi(u) = 2$, $\pi(v) \in \{3, 4\}$.
   If $\pi(v) = 3$ (resp. $\pi(v) = 4$), we necessarily have $\{\pi(y_1), \pi(y_2), \pi(y'_1), \pi(y'_2)\} = \{1, 4\}$ (resp. $\{\pi(y_1), \pi(y_2), \pi(y'_1), \pi(y'_2)\} = \{1, 3\}$), a contradiction since any two vertices in $\{y_1, y_2, y'_1, y'_2\}$ are linked by a directed path (in either direction) of length at most 3.

5. $\pi(u) = 3$, $\pi(v) = 4$.
   Since each vertex x_i, $1 \le i \le 4$, is linked by a directed path (in either direction) of length at most 3 to u and by a directed path (in either direction) of length at most 4 to v, we necessarily have $\{\pi(x_1), \pi(x_2), \pi(x_3), \pi(x_4)\} = \{1, 2\}$, a contradiction.

Therefore, every packing coloring of an oriented generalized theta graph containing Θ_0 as a subgraph must use 5 colors. ■

By Proposition 4, we know that for every oriented generalized theta graph Θ, $\chi_\rho(\Theta) = 2$ if and only if Θ is bipartite and one part of the bipartition contains only sources or sinks. However, characterizing oriented generalized theta graphs with packing chromatic number 3, 4 or 5 seems to be not so easy and we leave it as an open question.

From Lemma 11, we get that $\chi_\rho(\Theta) \le 4$ whenever Θ does not contain any path of length less than 5. However, this value of 5 cannot be decreased to 4 since we can construct oriented generalized theta graphs with no path of length less than 4 with packing chromatic number 5. We have for instance such an example with 17 paths of length 4, 5, 6 or 7.

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