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An effective approach for integer partitions using exactly two distinct sizes of parts

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Abstract: In this paper we consider the number of partitions of a positive integer n into parts of a specified number of distinct sizes. We give a method for constructing all partitions of n into parts of two sizes, as well as an explicit formula to count them with a new self-contained proof. As a side effect, by using the möbius function we also give a formula for the number of partitions of n into coprime parts.

Keywords: Integer partitions, partitions into parts of different sizes, partitions into parts of two sizes, divisors number, Möbius function.

1 Introduction

A partition of a positive integer n is a sequence of non increasing positive integers n_1 (a_1 times), n_2 (a_2 times), ..., n_s (a_s times), with $n_i > n_{i+1}$, that sum to n. We sometimes write the such partition $\pi = (n_1^{a_1}\ n_2^{a_2} \cdots\ n_s^{a_s})$, each n_i is called part of the partition π and a_i its frequency. The partition function $p(n)$ counts the partitions of n. If we ignore some unpublished work of G.W.V. Leibniz, the theory of integer partitions can find its origin in the work of L. Euler [6]. In fact, he made a sustained study of partitions and partition identities, and exploited them to establish a huge number of results in Analysis in 1748. An excellent introduction to this subject can be found in the book of G. E. Andrews [2].

Definition 1 *Let $\pi = (n_1^{a_1}\ n_2^{a_2} \cdots\ n_s^{a_s})$ be a partition of n. We say that π is a partition into k parts with s distinct sizes if*

$$\begin{cases} n = a_1\ n_1 + \cdots + a_s\ n_s; \\ n_1 > n_2 > \cdots > n_s \geq 1; \\ a_1 + \cdots + a_s = k; \\ a_1, \ldots, a_s \geq 1. \end{cases} \tag{1}$$

Let $t(n,k,s)$ be the number of solutions of system (1) and $t(n,s)$ the total number of partitions of n into s distinct sizes. Then we have

$$t(n,s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} t(n,k,s). \tag{2}$$

Example 1 *Among 27 partitions of $n = 11$ into 2 distinct sizes, the partitions $(7^1\ 1^4)$, $(4^2\ 1^3)$, $(3^1\ 2^4)$ and $(3^3\ 1^2)$ are the only ones which are into 5 parts.*

This kind of partitions appeared for the first time in the work of P. A. MacMahon [7]. Next, E. Deutsch presented the number of partitions of n into exactly two odd sizes of parts and the number of partitions of n into exactly two sizes of parts, one odd and one even. One can find these values in the Online Encyclopedia of Integer Sequences (OEIS) [8] as A117955 for the first number, A117956 for the second one and A002133 for the number of partitions of n using only 2 types of parts. In the work of Benyahia-Tani and Bouroubi [3], we can find proof of effective and non-effective finiteness theorems on $t(n,k,s)$. We can cite for example the following results:

Theorem 1 *For $k \geq s \geq 2$, $n \geq k + \frac{s(s-1)}{2}$ and $n \geq \max\{k, \frac{s(s+1)}{2}\}$, we have*

$$t(n,k,s) = \sum_{i=1}^{\lfloor \frac{2n-s(s-1)}{2k} \rfloor} \sum_{j=1}^{k-s+1} t(n-ki, k-j, s-1), \tag{3}$$

$$t(n,k,2) = \sum_{i=1}^{\lfloor \frac{n-1}{k} \rfloor} \tau_{k-1\downarrow}(n-ki), \tag{4}$$

where $\tau_{d\downarrow}(k)$ denotes the number of positive divisors of k less than or equal to d.

2 Main Results

One of the aim of this paper is to give an explicit formula for $t(n,k,2)$ using an effective new approach.

Thus, let consider the system:

$$\begin{cases} n = a_1\ n_1 + a_2\ n_2; \\ a_1 + a_2 = k; \\ n_1 > n_2 \geq 1; \\ a_1, a_2 \geq 1. \end{cases} \tag{5}$$

and let $m = n_1 - n_2$ throughout the remainder of the paper.

First of all, we introduce the following lemma to prepare the main theorem.

Lemma 2 *System (5) has integral solutions if and only if the following conditions are satisfied:*

(i) $n \equiv n_2 k (mod\ m)$,

(ii) max $\left(1, \lceil \frac{n}{k} \rceil - m + \chi(k|n)\right) \leq n_2 \leq \lfloor \frac{n}{k} \rfloor - \chi(k|n)$,

where $\chi(k|n) = 1$ if k divides n, and 0 otherwise.

Proof. From system (5), we have

$$\begin{pmatrix} n_1 & n_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}.$$

Since $m > 0$, we can write

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} n - n_2 k \\ -n + n_1 k \end{pmatrix}.$$

Then, system (5) has integral solutions if and only if m divides $n - n_2k$, $n - n_2k > 0$ and $-n + n_1k > 0$. That is,

$$n \equiv n_2 k (mod\ m) \quad \text{and} \quad \frac{n}{k} - m < n_2 < \frac{n}{k}\cdot$$

Since k can divide n, and $n_2 \geq 1$, the result holds. ■

From this lemma, we can now derive the following theorem.

Theorem 3 *For $k \geq 2$, $n \geq max\{k, 3\}$, $d = gcd(n, k)$ and $e|d$, let $\mathfrak{I}_{\mathfrak{e}}$ be the set of pairs $(\alpha, \beta) \in \mathbb{N}^2$, such that:*

- $1 \leq \alpha \leq \left\lfloor \frac{n-k}{e} \right\rfloor$ *and* $\gcd\left(\alpha, \frac{k}{e}\right) = 1$,
- $\beta \equiv \left(\frac{n}{e}\right) \left(\frac{k}{e}\right)^{-1} (mod\ \alpha)$ *and* $0 \leq \beta \leq min\left(\alpha - 1, \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n)\right)$.

Then

$$t(n, k, 2) = \sum_{e|d} \sum_{(\alpha,\beta)\in\mathfrak{I}_{\mathfrak{e}}} \left(\left\lfloor \frac{\left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor - \left\lceil \frac{max\left(1, \left\lceil \frac{n}{k} \right\rceil + \chi(k|n) - \alpha e\right) - \beta}{\alpha} \right\rceil + 1 \right).$$

Proof. Put $e = gcd(m, k)$ and let $\alpha = \frac{m}{e}$, that is $1 \leq \alpha \leq \left\lfloor \frac{n-k}{e} \right\rfloor$ and $gcd\left(\alpha, \frac{k}{e}\right) = 1$. By Lemma 2, case (i), we can see that e divides d, and $n_2 \equiv \left(\frac{n}{e}\right) \left(\frac{k}{e}\right)^{-1} (mod\ \alpha)$.

Let $0 \leq \beta < \alpha$, such that $\beta \equiv \left(\frac{n}{e}\right) \left(\frac{k}{e}\right)^{-1} (mod\ \alpha)$. Then

$$n_2 = \beta + t\alpha, \ t \in \mathbb{Z}.$$

Since $0 \leq \beta < \alpha$ and $\beta \leq n_2 > 0$, then $t \in \mathbb{N}$ and $0 \leq \beta \leq min\left(\alpha - 1, \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n)\right)$.

It follows from Lemma 2, case (ii), that

$$max\left(1, \left\lceil \frac{n}{k} \right\rceil + \chi(k|n) - m\right) \leq \beta + t\alpha \leq \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n)\cdot$$

Finally, $t(n, k, 2)$ equals the number of positive integers t, such that

$$\left\lceil \frac{max\left(1, \left\lceil \frac{n}{k} \right\rceil + \chi(k|n) - m\right) - \beta}{\alpha} \right\rceil \leq t \leq \left\lfloor \frac{\left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor.$$

This completes the proof. ■

Remark 4 *One nice application of Theorem 3 concerns the following algorithm which allows us to generate all partitions of n using exactly two distinct sizes of parts.*

Algorithm 1 Partitions into k parts with exactly two distinct sizes of parts

Require: $k \geq 2$, $n \geq max\{k, 3\}$
Ensure: Set of quadruple (n_1, a_1, n_2, a_2),

```
d ← gcd(n, k)
for each divisor e of d do
  for α from 1 to ⌊(n−k)/e⌋ do
    if gcd(α, k/e) = 1 then
      β ← (n/e)(k/e)^{−1} (mod α)
      if β ≤ min(α − 1, ⌊n/k⌋ − χ(k|n)) then
        t_1 ← ⌈(max(1, ⌈n/k⌉ + χ(k|n) − αe) − β)/α⌉
        t_2 ← ⌊(⌊n/k⌋ − χ(k|n) − β)/α⌋
        for t from t_1 to t_2 do
          n_2 ← β + tα
          n_1 ← αe + n_2
          a_2 ← ⌊(n − n_1 k)/(n_2 − n_1)⌋
          a_1 ← k − a_2
        end for
      end if
    end if
  end for
end for
```

This algorithm runs in $O(n)$.

Example 2 *Let $n = 11$ and $k = 8$, then $d = gcd(11, 8) = 1$. So, $e = 1$ is the only one divisor of d. The values of α that satisfies $1 \leq \alpha \leq 3$ and $gcd(\alpha, 8) = 1$ are 1 or 3.*

1. *For $\alpha = 1$, we get $\beta \equiv 11.8^{-1}(mod\ 1) = 0$, which is $\leq min(0, 1)$. The pair $(\alpha, \beta) = (1, 0)$ is then accepted and gives only one value of t:*

$$t = \left\lfloor \frac{1-0}{1} \right\rfloor - \left\lceil \frac{max(1, 2-1) - 0}{1} \right\rceil + 1 = 1.$$

Therefore, we have only one partition corresponding to the pair $(\alpha, \beta) = (1, 0)$. By applying Algorithm 4, we get:

$$n_2 = 1,\ n_1 = 2,\ a_2 = 5 \text{ and } a_1 = 3,$$

and thus the partition $(2^3\ 1^5)$.

2. *For* $\alpha = 3$, *we get* $\beta = 1 \leq min(2,1)$, *then the pair* $(\alpha, \beta) = (3,1)$ *is accepted and gives the values:*

$$t = 1,\ n_2 = 1,\ n_1 = 4,\ a_2 = 7 \text{ and } a_1 = 1.$$

Thus the associated partition is $(4^1\ 1^7)$.

We get, finally

$$t(11, 8, 2) = 2.$$

Example 3 *Let* $n = 22$ *and* $k = 8$, *then* $d = gcd(22, 8) = 2$. *So, we have two divisors of* d, $e = 1$ *and* $e = 2$.

- *Case1:* $e = 1$.
  The values of α *that satisfies* $1 \leq \alpha \leq 14$ *and* $gcd(\alpha, 8) = 1$ *are* $1, 3, 5, 7, 9, 11$ *or* 13.

  1. *For* $\alpha = 1$, *we get* $\beta = 0$. *The pair* $(1,0)$ *is accepted and gives the values:*

     $$t = 1,\ n_2 = 2,\ n_1 = 3,\ a_2 = 2 \text{ and } a_1 = 6,$$

     and then the partition $(3^6\ 2^2)$.
  2. *For* $\alpha = 3$, *we get* $\beta = 2$. *The pair* $(3,2)$ *is accepted and gives the values:*

     $$t = 1,\ n_2 = 2,\ n_1 = 5,\ a_2 = 6 \text{ and } a_1 = 2,$$

     and then the partition $(5^2\ 2^6)$.
  3. *For* $\alpha = 5$, *we get* $\beta = 4 > min(4,2)$, *then the pair* $(5,4)$ *is rejected.*
  4. *For* $\alpha = 7$, *we get* $\beta = 1$. *The pair* $(7,1)$ *is accepted and gives the values:*

     $$t = 1,\ n_2 = 1,\ n_1 = 8,\ a_2 = 6 \text{ and } a_1 = 2,$$

     and then the partition $(8^2\ 1^6)$.
  5. *For* $\alpha = 9$, *we get* $\beta = 5 > min(8,2)$, *then the pair* $(9,5)$ *is rejected.*
  6. *For* $\alpha = 11$, *we get* $\beta = 3 > min(10,2)$, *then the pair* $(11,3)$ *is rejected.*
  7. *For* $\alpha = 13$, *we have* $\beta = 6 > min(13,2)$, *then the pair* $(13,6)$ *is rejected.*

- *Case2:* $e = 2$.
  The values of α *that satisfies* $1 \leq \alpha \leq 7$ *and* $gcd(\alpha, 8) = 1$ *are* $1, 3, 5$ *or* 7.

  1. *For* $\alpha = 1$, *we have* $\beta = 0$. *The pair* $(1,0)$ *is accepted and gives* $1 \leq t \leq 2$. *Applying Algorithm 4, we obtain two partitions corresponding to the pair* $(1,0)$; *the first one is* $(3^7\ 1^1)$ *for* $t = 1$ *and the second one is* $(4^3\ 2^5)$ *for* $t = 2$.

2. *For $\alpha = 3$, we get $\beta = 2$. The pair $(3,2)$ is accepted and gives the values:*

$$t = 1,\ n_2 = 2,\ n_1 = 8,\ a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(8^1\ 2^7)$.

3. *For $\alpha = 5$, we get $\beta = 4 > min(4,2)$, the pair $(5,4)$ is rejected.*
4. *For $\alpha = 7$, we get $\beta = 1$. The pair $(7,1)$ is accepted and gives the values:*

$$t = 1,\ n_2 = 1,\ n_1 = 15,\ a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(15^1\ 1^7)$.

We get, finally

$$t(22,8,2) = 7.$$

After having counting the number $t(n,k,s)$, it would be of considerable interest to explore the number of partitions of n into k parts with exactly s distinct coprime sizes, which we denote by $g(n,k,s)$. Thus, let set

$$g(n,s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} g(n,k,s). \tag{6}$$

Theorem 5 *For $k \geq s \geq 2$ and $n \geq \max\{k, \frac{s(s+1)}{2}\}$, we have*

$$g(n,k,s) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\ t(d,k,s), \tag{7}$$

where $\mu(.)$ denotes Möbius function.

Proof. Let $T(n,k,s)$ be the set of partitions of n into k parts with s distinct sizes and $G(n,k,s)$ the subset of the such partitions but with s distinct coprimes sizes. We notice that, the mapping from the set $T(n,k,s)$ to $\bigcup_{d|n} G(d,k,s)$ defined by:

$$(n_1^{a_1}\ n_2^{a_2} \cdots\ n_s^{a_s}) \to \left(\left(\frac{n_1}{\delta}\right)^{a_1}\ \left(\frac{n_2}{\delta}\right)^{a_2} \cdots\ \left(\frac{n_s}{\delta}\right)^{a_s}\right),$$

is a bijection, where $\delta = gcd(n_1, n_2, \ldots, n_s)$.

Consequently, we have

$$t(n,k,s) = \sum_{d|n} g(d,k,s). \tag{8}$$

Hence, the result follows by using the Möbius inversion formula. ■

Remark 6 *Since $t(d,k,s) = 0$ if $d < \max\left\{k, \frac{s(s+1)}{2}\right\}$, the summation in (7) can be extended only over all divisors d of n such that $\frac{n}{d} \geq \max\left\{k, \frac{s(s+1)}{2}\right\}$. For example, if we take $n = 22$ and $k = 8$, then*

$$g(22,8,2) = \mu\,(2)\,t(11,8,2) + \mu\,(1)\,t(22,8,2),$$

and, according to Examples 2 and 3, we get $g(22,8,2) = 7 - 2 = 5$. These partitions are: $(3^7\ 1^1), (3^6\ 2^2), (5^2\ 2^6), (8^2\ 1^6)$ and $(15^1\ 1^7)$.

Using Theorems 5 and 3, we can construct the following table:

$n \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$g(n,2)$
3	1																		1
4	1	1																	2
5	2	2	1																5
6	1	1	2	1															5
7	3	3	2	2	1														11
8	2	2	2	2	2	1													11
9	3	3	2	3	2	2	1												16
10	2	2	4	1	3	2	2	1											17
11	5	5	3	4	2	3	2	2	1										27
12	2	2	2	2	3	2	3	2	2	1									21
13	6	6	4	5	2	4	2	3	2	2	1								37
14	3	3	5	3	4	1	4	2	3	2	2	1							33
15	4	4	3	3	4	4	2	4	2	3	2	2	1						38
16	4	4	5	3	4	3	3	2	4	2	3	2	2	1					42
17	8	8	5	7	3	5	3	4	2	4	2	3	2	2	1				59
18	3	3	5	2	5	2	4	2	4	2	4	2	3	2	2	1			46
19	9	9	6	7	3	7	3	4	3	4	2	4	2	3	2	2	1		71
20	4	4	4	4	4	3	6	2	3	3	4	2	4	2	3	2	2	1	57

Table 1: $g(n,k,2)$, $2 \leq k < n \leq 20$.

From identity (7) we can see that if $k \geq \lfloor \frac{n}{2} \rfloor$, then $t(n,k,2) = g(n,k,2)$. In the present theorem we present this observation in a more explicit form.

Theorem 7 *For $n \geq \max\{3,k\}$ and $k \geq \max\{2, \lfloor \frac{n}{2} \rfloor\}$, we have*

$$t(n,k,2) = g(n,k,2) = \tau(n-k) - \chi(n = 2k),$$

where $\tau(n)$ denotes the number of positive divisors of n and $\chi(n = 2k) = 1$ if $n = 2k$, 0 otherwise.

Proof. Let us first notice that if $k \geq 1 + \max\{2, \lfloor \frac{n}{2} \rfloor\}$, then $k \geq \lceil \frac{n+1}{2} \rceil$, and by Identity (4) the result yields (see [3], Corollary 3). Let now $k = \max\{2, \lfloor \frac{n}{2} \rfloor\}$. Since the result is true for $n = 3$, we can assume $k = \lfloor \frac{n}{2} \rfloor$. Let $\pi = (n_1^{a_1}\ n_2^{a_2})$ be a partition of n into k parts with two distinct sizes. If n is even, then $n_2 = 1$, else $n > (a_1 + a_2)\ n_2 = k\ n_2 \geq 2 \left\lfloor \frac{n}{2} \right\rfloor = n$, a contradiction. Hence, $n - k = (n_1 - 1)\ a_1$, in which case $n_1 - 1$ divides $n - k$. So, for each divisor d of $n - k$, we get $n_1 = d + 1, a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{d} = 0$. Thus, the result yields.

Now, if n is odd, then $n_2 = 1$ or $(n_1, n_2) = (3, 2)$. Indeed, if ($n_2 = 2$ and $n_1 \geq 4$) or ($n_2 \geq 3$), then $n > 3a_1 + 2a_2 = 2k + a_1 \geq 2 \left\lfloor \frac{n}{2} \right\rfloor + 1 = n$, a contradiction. In case of $n_2 = 1$, by the same argument above, we get for each divisor d of $n - k$, $n_1 = d + 1, a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{d} < 0$, which is completed by the partition $(3^{n-2k}\ 2^{3k-n})$. This completes the proof. ∎

Remark 8 *As shown in the proof above, the* $t(n, k, 2)$*'s partitions have been generated explicitly.*

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory.* Springer-Verlag, New York (1976).

[2] G. E. Andrews and K. Eriksson, *Integer Partitions.* Cambridge University Press, Cambridge (2004).

[3] N.B. Tani and S. Bouroubi, Enumeration of the partitions of an integer into parts of a specified number of different sizes and especially two sizes, Journal of Integer Sequences, 14, Article 11.3.6 (2011).

[4] A. Charalambos Charalambides, *Enumerative Combinatorics*, Chapman & Hall/CRC (2002).

[5] L. Comtet, *Advanced Combinatorics*, pp. 133–175. D. Reidel Publishing Company, Dordrecht-Holland, Boston, (1974).

[6] E. Knobloch, Leibniz on combinatorics, pp. 409–430. Historia Math, 1 (1974).

[7] P. A. MacMahon, Divisors of numbers and their continuations in the theory of partitions, *Proc. London Math. Soc.*, **2**, pp. 75–113 (1919).

[8] OEIS : The online Encyclopedia of Integer sequences. Published electronically at http://oeis.org, (2008).

[9] I. Pak, Partition bijections, a survey, *Ramanujan Journal*, **12**, 5–75 (2006).

[10] H. Rademacher, On the partition function $p(n)$, *Proc. London, Math. Soc.*, **43**, 241–254 (1937).

[11] H. S. Wilf, Lectures on integer partitions, unpublished, http://cis.upenn. edu / wilf.