Laboratoire d'Informatique Fondamentale, de Recherche Opérationnelle, de Combinatoire et d'Économétrie

Availaible on line at http://www.liforce.usthb.dz
bulletin-liforce@usthb.dz
Bulletin du Laboratoire 01 (2022) 01 - 07

# A closed formula for the number of ordered quadrilaterals with sides of integer length and fixed perimeter

**Sadek BOUROUBI**

[1,2] USTHB, Faculty of Mathematics,
P.B. 32 El-Alia, 16111, Bab Ezzouar, Algiers, Algeria.

sbouroubi@usthb.dz or bouroubis@gmail.com.

**Abstract :** Given a positive integer $n$, how many quadrilaterals with ordered integer sides and perimeter $n$ are there? Denoting the number of such quadrilaterals by $Q(n)$, the answer is given by:

$$Q(n) = \left\{ \frac{1}{576} n\,(n+3)\,(2n+3) - \frac{(-1)^n}{192} n\,(n-5) \right\}.$$

**Keywords:** Integer quadrilaterals, Integer partitions, Generating function.

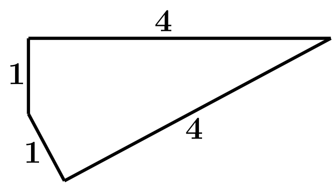
# 1 Introduction

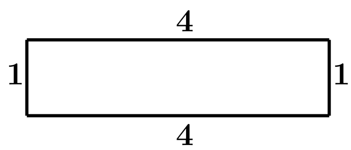
In [1], Jordan, Walch, and Wisner characterized the number $T(n)$ of incongruent triangles with integer sides that have a fixed perimeter $n$ by proving that $T(2n+12) = T(2n-3)+n+3$ for $n \geq 1$. However, in [2], George E. Andrews noted that $T(n)$ can be simply handled by relating it to $p_3(n)$ and $p_2(n)$, the number of partitions of $n$ into 3 and 2 parts, respectively, and proved the following analytical formula:

$$T(n) = \left\{\frac{n^2}{12}\right\} - \left\lfloor\frac{n}{4}\right\rfloor \left\lfloor\frac{n+2}{4}\right\rfloor,$$

where $\{x\}$ is the nearest integer function and $\lfloor x \rfloor$ the greatest integer function.

In the following, we will deal with the same problem, but by regarding the number $Q(n)$ of incongruent quadrilaterals with integer sides and perimeter $n$, which have the sequence of their sides ordered, which we just call ordered integer quadrilaterals. For example, the 4-tuple $(1,1,4,4)$ of perimeter $n=10$ is ordered; it can be rearranged to generate the unordered 4-tuple $(1,4,1,4)$, so that the first forms a kite and the second a rectangle as shown below:




# 2 Preliminary results

The partition of $n \in \mathbb{N}$ into $k$ parts is a tuple $\pi = (\pi_1, \ldots, \pi_k) \in \mathbb{N}^k, k \in \mathbb{N}$, such that:

$$n = \pi_1 + \cdots + \pi_k,\ 1 \leq \pi_1 \leq \cdots \leq \pi_k,$$

where the nonnegative integers $\pi_i$ are called parts. We denote the number of partitions of $n$ into $k$ parts by $p(n,k)$.

**Lemma 1** *For $n \geq 4$, we have:*

$$Q(n) = p(n,4) - \sum_{m=3}^{\left\lfloor \frac{n}{2} \right\rfloor} p(m,3) \cdot$$

**Proof.** At first sight, it should be noted that any partition of $n$ into four parts generates an ordered integer quadrilateral and vice versa, except the partitions for which the sum of its three small parts does not exceed the largest part, due to the triangle inequality, the such partitions verify:

$$n = a + b + c + d, \quad 1 \le a \le b \le c \le d \ \text{ and } \ a + b + c \le d,$$

or

$$n - d = a + b + c, \quad 1 \le a \le b \le c \le d \le n.$$

But

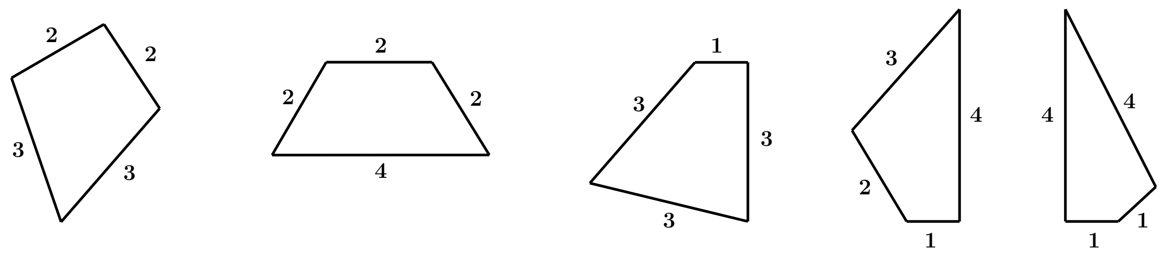
$$n - d \le d \Longleftrightarrow n - d \le \frac{n}{2}\cdot$$

Hence

$$Q(n) = p\,(n, 4) - \sum_{m=3}^{\left\lfloor \frac{n}{2} \right\rfloor} p\,(m, 3)\,\cdot$$

■

If we consider, for example, the perimeter $n = 10$, then the number of partitions of $n$ is 9, which are: 7111, 6211, 5311, <u>4411</u>, 5221, <u>4321</u>, <u>3331</u>, <u>4222</u> and <u>3322</u>, they form a quadrilateral only those we have underlined as shown below:




As we can check:

$$Q(10) = p\,(10, 4) - \sum_{m=3}^{5} p(m, 3) = 9 - (1 + 1 + 2) = 5.$$

It should be noted that each quadrilaterals in the figure above represents an equivalence class of quadrilaterals that all share the same partition. So, the number $Q(n)$ counts only the incongruent ordered integer quadrilaterals representing the equivalence classes modulo the same partition.

**Lemma 2** *For $n \ge 3$, we get:*

$$\sum_{m=3}^{n} p\,(m, 3) = \frac{n\,(n-2)\,(2n+7)}{72} + \frac{1}{3}\left\lfloor \frac{n}{3} \right\rfloor + \frac{1 - (-1)^n}{16}\cdot$$

**Proof.** Let $f(z)$ be the known generating function of $p(m,3)$ [3]:

$$f\left(z\right)=\frac{z^3}{\left(1-z\right)\left(1-z^2\right)\left(1-z^3\right)}.$$

Then

$$\sum_{m=0}^{n}p(m,3)=\left[z^n\right]\left(\frac{f\left(z\right)}{1-z}\right).$$

From expanding $\dfrac{f\left(z\right)}{1-z}$ in partial fractions, we obtain:

$$\frac{f\left(z\right)}{1-z}=\frac{1}{36}\frac{z^3+4z^2+z}{\left(1-z\right)^4}+\frac{1}{24}\frac{z^2+z}{\left(1-z\right)^3}-\frac{1}{12}\frac{z}{\left(1-z\right)^2}-\frac{1}{17}\frac{1}{1-z}-\frac{1}{16}\frac{1}{1+z}+\frac{1}{9}\frac{1+z}{1+z+z^2}.$$

Since

$$\begin{aligned}
\frac{1}{1-z} &= \sum_{n\geq 0} z^n,\\
\frac{1}{1+z} &= \sum_{n\geq 0} \left(-1\right)^n z^n,\\
\frac{z}{\left(1-z\right)^2} &= \sum_{n\geq 0} nz^n,\\
\frac{z^2+z}{\left(1-z\right)^3} &= \sum_{n\geq 0} n^2 z^n,\\
\frac{z^3+4z^2+z}{\left(1-z\right)^4} &= \sum_{n\geq 0} n^3 z^n,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1+z}{1+z+z^2} &= \frac{1-z^2}{1-z^3},\\
&= \frac{1}{1-z^3}-x^2\frac{1}{1-z^3},\\
&= \sum_{n\geq 0} z^{3n}-\sum_{n\geq 0} z^{3n+2},\\
&= \sum_{n\geq 0} a_n z^n,
\end{aligned}$$

where

$$a_n=\begin{cases} 1, & n\equiv 0 \pmod 3,\\ 0, & n\equiv 1 \pmod 3,\\ -1, & n\equiv 2 \pmod 3. \end{cases}$$

In a simplified way,

$$a_n=1-n+3\left\lfloor\frac{n}{3}\right\rfloor.$$

Summing all coefficients of $z^n$, the result yields. ∎

**Corollary 3** *For $n \geq 6$, we have:*

$$\sum_{m=3}^{\left\lfloor \frac{n}{3} \right\rfloor} p\left(n,3\right) = \frac{1}{576}\left(2n^3+3n^2-59n+30\right)+\frac{\left(-1\right)^n}{192}\left(n^2+n-10\right)+\frac{1}{3}\left\lfloor \frac{n}{6} \right\rfloor+\frac{1-\left(-1\right)^{\left\lfloor \frac{n}{2} \right\rfloor}}{16}\cdot$$

**Proof.** While observing that

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}-\frac{1-\left(-1\right)^n}{4},$$

we get, from Lemma 2,

$$\frac{1}{72}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor \left(2\left\lfloor \frac{n}{2} \right\rfloor+7\right) = \frac{1}{576}\left(2n^3+3n^2-59n+30\right)+\frac{\left(-1\right)^n}{192}\left(n^2+n-10\right)\cdot$$

Hence, the result follows. ■

# 3 Main result

**Theorem 4** *For $n \geq 4$, we have:*

$$Q(n) = \left\{\frac{1}{576}n\left(n+3\right)\left(2n+3\right)-\frac{\left(-1\right)^n}{192}n\left(n-5\right)\right\}\cdot$$

**Proof.** The generating function of $p(n,4)$ is as follows [3]:

$$g\left(z\right) = \frac{z^4}{\left(1-z\right)\left(1-z^2\right)\left(1-z^3\right)\left(1-z^4\right)}\cdot$$

Via straightforward calculations, it can be proved that

$$p\left(n,4\right) = \frac{n^3}{144}+\frac{n^2}{48}-\frac{\left(1-\left(-1\right)^n\right)n}{32}+\frac{\left(-1\right)^n}{32}-\frac{13}{288}+\frac{\alpha_n}{72},$$

where

$$\alpha_n \in \left\{-17,-9,-8,-1,0,1,8,9,17\right\}\cdot$$

Then, from Lemma 1 and Corollary 3, we get:

$$Q_n = \frac{1}{576}n\left(n+3\right)\left(2n+3\right)-\frac{\left(-1\right)^n}{192}n\left(n-5\right)+\beta_n,$$

where

$$\beta_n = -\frac{23}{144}+\frac{\left(-1\right)^{\left\lfloor \frac{n}{2} \right\rfloor}}{16}+\frac{1}{3}\left(\frac{n}{6}-\left\lfloor \frac{n}{6} \right\rfloor\right)+\frac{\left(-1\right)^n}{12}-\frac{\alpha_n}{72},$$

with

$$\beta_n \in \left\{-\frac{3}{8}, -\frac{1}{4}, -\frac{11}{72}, -\frac{5}{36}, -\frac{1}{8}, -\frac{1}{36}, -\frac{1}{72}, 0, \frac{5}{72}, \frac{7}{72}, \frac{2}{9}, \frac{4}{9}\right\}.$$

Since $Q(n)$ is an integer and $|\beta(n)| < 1/2$, we finally get:

$$Q(n) = \left\{\frac{1}{576} n(n+3)(2n+3) - \frac{(-1)^n}{192} n(n-5)\right\}.$$

■

By using a computer algebra package, Theorem 4 allows us to obtain $Q(n)$ for large some values of $n$. The following table is introduced to illustrate a few:

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Q(n))$ | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 11 | 12 | 16 | 18 | 23 | 24 | 31 | 33 |

**Remark 5** *It is easy to see, from Theorem 4, that for $n \geq 1$, we get:*

$$Q(24n) = 48n^3 + 6n^2 + n.$$

# 4 Conclusion

The values $Q(n)$ in the table above are sequence A062890 in the Online Encyclopedia of Integer Sequences [8], but no explicit formula has been given for this sequence.

# 5 Acknowledgment

The author would like to thank Professeur Farid Bencherif for his careful reading of the manuscript and for his valuable remarks.

# References

[1] J. H. Jordan, R. Walch, and R. J. Wisner, Triangles with integer sides, Notices Amer. Math. Soc., 24 (1977) A-450.

[2] George E. Andrews (1979) A Note on Partitions and Triangles with Integer Sides, The American Mathematical Monthly, 86:6, 477-478, DOI: 10.1080/00029890.1979.11994832

[3] G. E. Andrews and K. Eriksson, 'Integer Partitions', Cambridge University Press, Cambridge, United Kingdom, (2004).

[4] Anning, N., Frame, J. S., and Auluck, F. C. : Problems for Solution: 3874 - 3899. The American Mathematical Monthly, Vol. 47, No. 9, (Nov., 1940), 664 - 666

[5] Benyahia Tani, Nesrine, Yahi, Zahra, et Bouroubi, Sadek. Ordered and inordered incongruent convex quadrilaterals inscribed in a regular n-gon. Ian Schindler, 2011, vol. 66, p. 71.

[6] Bouroubi, Sadek, sur quelques relations relatives aux nombres des partitions d'un entier Maghreb, Math. Rev., Vol. 11, No 1, June, 2002, pp. 14-19.

[7] Bouroubi, Sadek, (2021). La Combinatoire et ses Applications. Pages bleues, Mitidja impression, Beraki, Alger.

[8] The Online Encyclopedia of Integer Sequences (2018), published electronically at http://oeis.org/.