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On the Broadcast Independence Number of Caterpillars

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Abstract: Let G be a simple undirected graph.A broadcast on G is a function $f : V(G) \to \mathbb{N}$ such that $f(v) \leq e_G(v)$ holds for every vertex v of G, where $e_G(v)$ denotes the eccentricity of v in G, that is, the maximum distance from v to any other vertex of G. The cost of f is the value $\mathrm{cost}(f) = \sum_{v \in V(G)} f(v)$. A broadcast f on G is independent if for every two distinct vertices u and v in G, $d_G(u, v) > \max\{f(u), f(v)\}$, where $d_G(u, v)$ denotes the distance between u and v in G. The broadcast independence number of G is then defined as the maximum cost of an independent broadcast on G. In this paper, we study independent broadcasts of caterpillars and give an explicit formula for the broadcast independence number of caterpillars having no pair of adjacent vertices with degree 2.

Keywords: Independence; Distance; Broadcast independence; Caterpillar.

1 Introduction

All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively.

For any two vertices u and v of G, the *distance* $d_G(u,v)$ between u and v in G is the length (number of edges) of a shortest path joining u and v. The *eccentricity* $e_G(v)$ of a vertex v in G the maximum distance from v to any other vertex of G. The minimum eccentricity in G is the *radius* $\mathrm{rad}(G)$ of G, while the maximum eccentricity in G is the *diameter* $\mathrm{diam}(G)$ of G. Two vertices u and v with $d_G(u,v) = \mathrm{diam}(G)$ are said to be *antipodal*.

A function $f : V(G) \to \{0, \dots, \mathrm{diam}(G)\}$ is a *broadcast* if for every vertex v of G, $f(v) \le e_G(v)$. The value $f(v)$ is called the *f-value of v*. Given a broadcast f on G, an *f-broadcast vertex* is a vertex v with $f(v) > 0$. The set of all f-broadcast vertices is denoted V_f^+. If $u \in V_f^+$ is a broadcast vertex, $v \in V(G)$ and $d_G(u,v) \le f(u)$, we say that u *f-dominates* v. In particular, every f-broadcast vertex f-dominates itself. The *cost* $\mathrm{cost}(f)$ of a broadcast f on G is given by

$$\mathrm{cost}(f) = \sum_{v \in V(G)} f(v) = \sum_{v \in V_f^+} f(v).$$

A broadcast f on G is a *dominating broadcast* if every vertex of G is f-dominated by some vertex of V_f^+. The minimum cost of a dominating broadcast on G is the *broadcast dominating number* of G, denoted $\gamma_b(G)$. A broadcast f on G is an *independent broadcast* if every f-broadcast vertex is f-dominated only by itself. The maximum cost of an independent broadcast on G is the *broadcast independence number* of G, denoted $\beta_b(G)$. An independent broadcast on G with cost β is an independent *β-broadcast*. An independent $\beta_b(G)$-broadcast on G is an *optimal* independent broadcast. Note here that any optimal independent broadcast is necessarily a dominating broadcast.

The notions of broadcast domination and broadcast independence were introduced by D.J. Erwin in his Ph.D. thesis [9] under the name of *cost domination* and *cost independence*, respectively. During the last decade, broadcast domination has been investigated by several authors, see e.g. [1, 2, 3, 5, 6, 7, 11, 12, 13, 14, 15, 16], while independent broadcast domination has attracted much less attention.

In particular, Seager considered in [15] broadcast domination of caterpillars. She characterized caterpillars with broadcast domination number equal to their domination number, and caterpillars with broadcast domination number equal to their radius. Blair, Heggernes, Horton and Manne proposed in [1] an $O(nr)$-algorithm for computing the broadcast domination number of a tree of order n with radius r.

However, determining the independent broadcast number of trees seems to be a difficult problem. We propose in this paper a first step in this direction, by studying a subclass of the class of caterpillars. Recall that a caterpillar is a tree such that deleting all its pendent vertices leaves a simple path. The subclass we will consider is the subclass of caterpillars having no pair of adjacent vertices with degree 2.

We now review a few results on independent broadcast numbers. Let G be a graph and $A \subset V(G)$, $|A| \geq 2$, be a set of pairwise antipodal vertices in G. The function f defined by $f(u) = \mathrm{diam}(G) - 1$ for every vertex $u \in A$, and $f(v) = 0$ for every vertex $v \notin A$, is clearly an independent $|A|(\mathrm{diam}(G) - 1)$-broadcast on G.

Observation 1 (Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [8])
For every graph G of order at least 2 and every set $A \subset V(G)$, $|A| \geq 2$, of pairwise antipodal vertices in G, $\beta_b(G) \geq |A|(\mathrm{diam}(G) - 1)$. In particular, for every tree T, $\beta_b(T) \geq 2(\mathrm{diam}(G) - 1)$.

An independent broadcast f on a graph G is *maximal* if there is no independent broadcast $f' \neq f$ such that $f'(v) \geq f(v)$ for every vertex $v \in V(G)$. In [9], D.J. Erwin proved the following result (see also [8]).

Theorem 2 (Erwin [9])
Let f be an independent broadcast on G. If $V_f^+ = \{v\}$, then f is maximal if and only if $f(v) = e_G(v)$. If $|V_f^+| \geq 2$, then f is maximal if and only if the following two conditions are satisfied:

1. *f is dominating, and*
2. *for every $v \in V_f^+$, $f(v) = \min\left\{d_G(v,u) :\ u \in V_f^+ \setminus \{v\}\right\} - 1$.*

Erwin proved that $\beta_b(P_n) = 2(n-2) = 2(\mathrm{diam}(P_n) - 1)$ for every path P_n of length $n \geq 3$ [9]. In [4], Bouchemakh and Zemir determined the independent broadcast number of square grids.

Theorem 3 (Bouchemakh and Zemir [4])
Let $G_{m,n}$ denote the square grid with m rows and n columns, $m \geq 2$, $n \geq 2$. We then have:

1. *$\beta_b(G_{m,n}) = 2(m+n-3) = 2(\mathrm{diam}(G_{m,n}) - 1)$ if $m \leq 4$,*
2. *$\beta_b(G_{5,5}) = 15$, $\beta_b(G_{5,6}) = 16$, and*
3. *$\beta_b(G_{m,n}) = \left\lceil \frac{mn}{2} \right\rceil$ for every m, n, $5 \leq m \leq n$, $(m,n) \neq (5,5), (5,6)$.*

In this paper, we determine the broadcast independence number of caterpillars having no pair of adjacent vertices with degree 2.

The paper is organized as follows. We introduce in the next section the main definitions and a few preliminary results on independent broadcasts of caterpillars. We then consider in Section 3 the case of caterpillars having no pair of adjacent vertices with degree 2 and prove our main result, which gives an explicit formula for the broadcast independence number of such caterpillars.

2 Preliminaries

Let G be a graph and H be a subgraph of G. Since $d_H(u,v) \geq d_G(u,v)$ for every two vertices $u, v \in V(H)$, every independent broadcast f on G satisfying $f(u) \leq e_H(u)$ for every vertex $u \in V(H)$ is an independent broadcast on H. Hence we have:

Observation 4 *If H is a subgraph of G and f is an independent broadcast on G satisfying $f(u) \leq e_H(u)$ for every vertex $u \in V(H)$, then the restriction f_H of f to $V(H)$ is an independent broadcast on H.*

A *caterpillar of length $k \geq 0$* is a tree such that removing all leaves gives a path of length k, called the *spine*. Following the terminology of [15], a non-leaf vertex is called a *spine vertex* and, more precisely, a *stem* if it is adjacent to a leaf and a *trunk* otherwise. A leaf adjacent to a stem v is a *pendent neighbor* of v.

Note that a caterpillar of length 0 is nothing but a star $K_{1,n}$, for some $n \geq 1$. The independent broadcast number of a star is easy to determine.

Observation 5 *For every integer $n \geq 1$, $\beta_b(K_{1,n}) = n$.*

Indeed, an optimal broadcast f of $K_{1,n}$ is obtained by setting to 1 the f-value of every pendent vertex of $K_{1,n}$, if $n > 1$, or of one of the two vertices of $K_{1,1}$. Therefore, in the rest of the paper, we will only consider caterpillars of length $k \geq 1$.

We denote by $CT(\lambda_0, \dots, \lambda_k)$, $k \geq 1$, with $(\lambda_0, \dots, \lambda_k) \in \mathbb{N}^* \times \mathbb{N}^{k-1} \times \mathbb{N}^*$, the caterpillar of length $k \geq 1$ with spine $v_0 \dots v_k$ such that each spine vertex v_i has λ_i pendent neighbors. Note that for any caterpillar CT of length $k \geq 1$, $\mathrm{diam}(CT) = k+2$. For every i such that $\lambda_i > 0$, $0 \leq i \leq k$, we denote by $\ell_i^1, \dots, \ell_i^{\lambda_i}$ the pendent neighbors of v_i. Moreover, we denote by $CT[a,b]$, $0 \leq a \leq b \leq k$, the subgraph of CT induced by vertices $v_a, \dots, v_b$ and their pendent neighbors The caterpillar $CT(1,0,2,1,1,2,1,0,3)$ is depicted in Figure 1. Let f be an independent broadcast on a caterpillar $CT = CT(\lambda_0, \dots, \lambda_k)$. We denote by

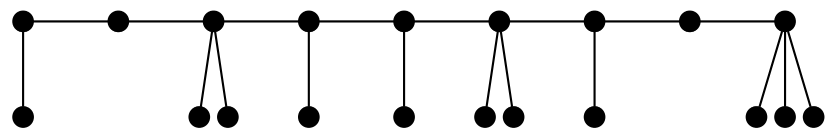


Figure 1: The caterpillar $CT(1,0,2,1,1,2,1,0,3)$

f^* the associated mapping from $\{v_0,\dots,v_k\}$ to $\mathbb{N}$ defined by

$$f^*(v_i) = f(v_i) + \sum_{j=1}^{j=\lambda_i} f(\ell_i^j),\ \text{if } \lambda_i > 0, \quad \text{and} \quad f^*(v_i) = f(v_i) \text{ otherwise,}$$

for every i, $0 \le i \le k$. Intuitively speaking, when $\lambda_i > 0$, $f^*(v_i)$ gives the "weight" of the star-graph consisting of the vertex v_i together with its pendent neighbors.

We will say that two independent broadcasts f_1 and f_2 on CT are *similar* whenever $f_1^* = f_2^*$. Observe that any two similar independent broadcast have the same cost.

From Observation 1, we get that $\beta_b(CT) \ge 2(k+1)$ for every caterpillar $CT = CT(\lambda_0,\dots,\lambda_k)$. In particular, the function f_c on $V(CT)$ defined by $f_c(\ell_0^1) = f_c(\ell_k^1) = k+1$ and $f_c(u) = 0$ for every vertex $u \in V(CT)\setminus\{\ell_0^1,\ell_k^1\}$ is an independent broadcast on CT with cost $2(k+1)$.

In the following, we will call any independent broadcast f similar to f_c and such that $|V_f^+| = 2$ a *canonical* independent broadcast.

The following lemma shows that, for any caterpillar $CT = CT(\lambda_0,\dots,\lambda_k)$, no independent broadcast f on CT with $f(v) > 0$ for some stem v can be optimal.

Lemma 6 *If $CT = CT(\lambda_0,\dots,\lambda_k)$ is a caterpillar of length $k \ge 1$ and f is an independent broadcast on CT with $f(v_i) > 0$ for some stem v_i, $0 \le i \le k$, then there exists an independent broadcast f' on CT with $\mathrm{cost}(f') > \mathrm{cost}(f)$.*

Proof. Since $f(v_i) > 0$ and f is an independent broadcast, we have $f(\ell_i^j) = 0$ for every j, $1 \le j \le \lambda_i$. Consider the function f' defined by $f'(v_i) = 0$, $f'(\ell_i^1) = f(v_i) + 1$ and $f'(u) = f(u)$ for every vertex $u \in V(CT) \setminus \{v_i, \ell_i^1\}$. Since $d_{CT}(\ell_i^1, u) = d_{CT}(v_i, u) + 1$ for every vertex $u \in V(CT) \setminus \{\ell_i^1\}$, we get that f' is an independent broadcast on CT. Moreover, we clearly have $\mathrm{cost}(f') = \mathrm{cost}(f) + 1$. □

The following lemma shows that for every optimal independent broadcast on a caterpillar, at least one pendent vertex of each of the end-vertices of the spine is a broadcast vertex.

Lemma 7 *Let $CT = CT(\lambda_0,\dots,\lambda_k)$ be a caterpillar of length $k \ge 1$. If f is an optimal independent broadcast on CT, then $f^*(v_0) - f(v_0) \ne 0$ and $f^*(v_k) - f(v_k) \ne 0$.*

Proof. Suppose to the contrary that $f(\ell_0^j) = 0$ for every j, $1 \leq j \leq \lambda_0$. We know by Lemma 6 that $f(v_0) = 0$. Let u be the f-broadcast vertex that dominates ℓ_0^1 and let $f(u) = x$. By Lemma 6, u is either a leaf or a trunk.

If u is a leaf, say $u = \ell_i^j$, $1 \leq i \leq k$, $1 \leq j \leq \lambda_i$, let f' be the mapping defined by $f'(\ell_0^1) = x + i$, $f'(u) = 0$ and $f'(u') = f(u')$ for every vertex $u' \in V(CT) \setminus \{\ell_0^1, u\}$. Note that every vertex which was f-dominated by u is now f'-dominated by ℓ_0^1. The mapping f' is thus an independent $(\mathrm{cost}(f) + i)$-broadcast on CT, contradicting the optimality of f.

If u is a trunk, say $u = v_i$, $1 \leq i \leq k - 1$, we similarly define a mapping f' by letting $f'(\ell_0^1) = x + i + 1$, $f'(u) = 0$ and $f'(u') = f(u')$ for every vertex $u' \in V(CT) \setminus \{\ell_0^1, u\}$. The mapping f' is thus an independent $(\mathrm{cost}(f) + i + 1)$-broadcast on CT, again contradicting the optimality of f.

The case $f(\ell_k^j) = 0$ for every j, $1 \leq j \leq \lambda_k$, follows by symmetry. □

Observe that Lemma 7 can be extended to trees as follows:

Lemma 8 *Let T be tree and T' be a subtree of T, of order at least 2, with root r. Let f be an optimal independent broadcast on T. If r is an f-broadcast vertex, then T' contains at least one other f-broadcast vertex. In particular, if T' is a subtree of height 1 (that is, $e_{T'}(r) = 1$), then $f(r) = 0$.*

Proof. Suppose to the contrary that $f(r) > 0$ and $f(u) = 0$ for every vertex $u \in V(T') \setminus \{r\}$. Let $t' = e_{T'}(r)$ and $\overline{t'} = e_{T-(T'-r)}(r)$.

If $f(r) < t'$, the independent broadcast f' given by $f'(v) = f(r)$ for some vertex v in T' with $d_{T'}(r, v) = t'$ and $f'(u) = f(u)$ for every vertex $u \in V'(T) \setminus \{v\}$ is such that $\mathrm{cost}(f') = \mathrm{cost}(f) + f(r)$, contradicting the optimality of f.

If $f(r) \geq \overline{t'}$, then r is the unique f-broadcast vertex, which implies $\mathrm{cost}(f) < 2(\mathrm{diam}(T) - 1)$, again contradicting the optimality of f by Observation 1.

Hence $\overline{t'} > f(r) \geq t'$. Let now v be any neighbor of r in T'. Since $\overline{t'} > f(r) \geq t'$, we have $e_T(v) = e_T(r) + 1 = \overline{t'} + 1 > f(r) + 1$. The function f' defined by $f'(r) = 0$, $f'(v) = f(r) + 1$ and $f'(u) = f(u)$ for every vertex $u \in V(T) \setminus \{r, v\}$ is therefore an independent broadcast on T with $\mathrm{cost}(f') = \mathrm{cost}(f) + 1$, contradicting the optimality of f.

This completes the proof. □

3 Caterpillars with no pair of adjacent trunks

In this section we determine the broadcast independence number of caterpillars with no pair of adjacent trunks. We first introduce some notation and useful lemmas.

We say that an independent broadcast f of a caterpillar CT is an *optimal non-canonical* independent broadcast on CT if

(i) $|V_f^+| \neq 2$ or $f^* \neq f_c^*$ (f is non-canonical), and

(ii) for every independent broadcast f' on CT with $|V_{f'}^+| \neq 2$ or $f'^* \neq f_c^*$, $\mathrm{cost}(f) \geq \mathrm{cost}(f')$ (f is optimal among all non-canonical independent broadcasts).

Let $CT = CT(\lambda_0, \dots, \lambda_k)$ be a caterpillar of length $k \geq 1$ with no pair of adjacent trunks. We denote by

$$\lambda(CT) = \sum_{i=0}^{i=k} \lambda_i$$

the number of leaves of CT, and by

$$\tau(CT) = |\{i \mid 1 \leq i \leq k-1 \text{ and } \lambda_i = 0\}|$$

the number of trunks of CT.

We will compute the broadcast independence number of a caterpillar with no pair of adjacent trunks by counting the number of some specific *patterns*. More precisely, we say that a pattern of length $p+1$, $\Pi = \pi_0 \dots \pi_p$, $p \geq 0$, $\pi_i \in \mathbb{N}$ for every i, $0 \leq i \leq p$, *occurs* in a caterpillar $CT = CT(\lambda_0, \dots, \lambda_k)$ if there exists an index i_0, $0 \leq i_0 \leq k-p$, such that $CT[i_0, i_0+p] = CT(\pi_0, \dots, \pi_p)$, that is, $\lambda_{i_0+j} = \pi_j$ for every j, $0 \leq j \leq p$. We will also say that the caterpillar CT *contains* the pattern Π and that the subgraph $CT(\lambda_{i_0}, \dots, \lambda_{i_0+p})$ of CT is an *occurrence* of the pattern Π. For instance, the caterpillar $CT(1,0,2,1,1,2,1,0,3)$, depicted on Figure 1, contains once the pattern 211 and twice the pattern 10.

We now extend the notation for patterns as follows:

- By π_i^+, we mean a spine vertex having at least π_i pendent neighbors;
- By π_i^-, we mean a spine vertex having at most π_i pendent neighbors;
- By $[\pi_i$, we mean that the leftmost stem, v_0, has π_i pendent neighbors (therefore, a pattern starting with this symbol must occur on the left end of a caterpillar);
- By $\pi_i]$, we mean that the rightmost stem, v_k, has π_i pendent neighbors (therefore, a pattern ending with this symbol must occur on the right end of a caterpillar);

- By $\{\pi_i, [\}\Pi$ (resp. $\Pi\{\pi_i,]\}$), we mean either the pattern $\pi_i\Pi$ (resp. $\Pi\pi_i$) or the pattern $[\Pi$ (resp. $\Pi]$),
- By $\pi_0(\pi_1\pi_2)^{+r}\pi_3$, we mean a *maximal* pattern of the form

$$\pi_0\pi_1\pi_2\pi_3 \quad \text{or} \quad \pi_0 \underbrace{\pi_1\pi_2 \dots \pi_1\pi_2}_{r \text{ times}, \ r \geq 2} \pi_3,$$

  where maximal here means that the subpattern $\pi_1\pi_2$ is repeated at least once and as many times as possible.

- By $\pi_0(\pi_1\pi_2)^{*r}\pi_3$, we mean a *maximal* pattern of the form

$$\pi_0\pi_3, \ \pi_0\pi_1\pi_2\pi_3 \quad \text{or} \quad \pi_0 \underbrace{\pi_1\pi_2 \dots \pi_1\pi_2}_{r \text{ times}, \ r \geq 2} \pi_3,$$

  where maximal here means that the subpattern $\pi_1\pi_2$ is repeated as many times as possible.

We can also combine these notations, so that $\pi_i^+]$, for instance, denotes that the rightmost stem v_k has at least π_i pendent neighbors.

One can check that the caterpillar $CT(1, 0, 2, 1, 1, 2, 1, 0, 3)$ contains once each of the patterns $[1,\ 3]$, $2^+]$ and 2111^+, twice the pattern $0\{2, 3\}$, and thrice the pattern $1^+1^+1^+$. On one other hand, the caterpillar $CT(1, 0, 2, 0, 2, 0, 2, 1, 0, 3)$ contains only once the pattern $1^+0(20)^{+r}1^+$, namely on the sub-caterpillar $CT(1, 0, 2, 0, 2, 0, 2)$ with explicit pattern 1020202.

For any pattern Π and any caterpillar CT, we will denote by $\#_{CT}(\Pi)$ the number of occurrences of the pattern Π in CT. Moreover, if M is an occurrence of Π in CT, we define the value

$$\alpha_1(M) = \max\{0, \#_M(1) - 1\},$$

that is, the number of stems v_i in M with $\lambda_i = 1$ minus 1—or 0 if M_1 contains no such stem—, and the value

$$\alpha_2(M) = \alpha_1(M) + \#_M([1^+) + \#_M(1^+]),$$

that is, $\alpha_1(M)$ plus 0, 1 or 2, depending on whether M contains no end-vertex of CT, one end-vertex of CT or both end-vertices of CT, respectively.

We then extend the functions α_1 and α_2 to the whole caterpillar CT by setting

$$\alpha_1(CT; \Pi) = \sum_{M \text{ occurrence of } \Pi} \alpha_1(M)$$

and

$$\alpha_2(CT; \Pi) = \sum_{M \text{ occurrence of } \Pi} \alpha_2(M).$$

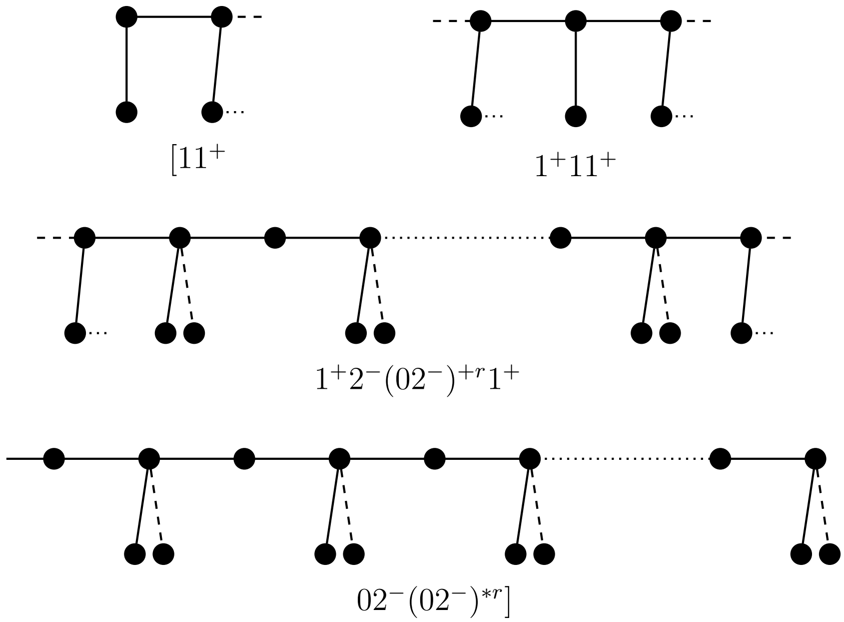


Figure 2: Sample patterns involved in the definition of $\beta^*(CT)$

Finally, for any caterpillar CT, we define the value $\beta^*(CT)$ as follows:

$$\begin{aligned}\beta^*(CT) \;=\;& \lambda(CT) + \tau(CT) + \#_{CT}(\{1^+,[\}1\{1^+,]\}) + \alpha_1(CT; 1^+2^-(02^-)^{+r}1^+) \\ & + \alpha_2(CT; 02^-(02^-)^{*r}0) + \alpha_2(CT; [2^-(02^-)^{*r}0) + \alpha_2(CT; 02^-(02^-)^{*r}]).\end{aligned}$$

Sample patterns involved in the above formula are illustrated on Figure 2. A pattern with a line to the left or right hand side of its spine cannot occur at the left or right end of the caterpillar, respectively. A pattern with a dashed line to the left or right hand side of its spine can occur at the left or right end of the caterpillar, respectively, or in the middle of the caterpillar. A dashed edge is an optional edge (used for pattern 2^-, corresponding to a spine vertex with either one or two pendent neighbors).

Let us say that two *distinct* occurrences of patterns *overlap* if they share a common vertex. Due to the specific structure of the patterns used in the above formula (and, in particular, of the maximality of the number of repetitions of subpatterns of the form Π^{+r} or Π^{*r}), we have the following:

Observation 9 *In every caterpillar CT of length $k \geq 1$,*

1. *no occurrence of the pattern $02^-(02^-)^{*r}0$ can overlap with an occurrence of a pattern $\{1^+,[\}1\{1^+,]\}$, $1^+2^-(02^-)^{+r}1^+$, $02^-(02^-)^{*r}0$, $[2^-(02^-)^{*r}0$ or $02^-(02^-)^{*r}]$,*
2. *no occurrence of the pattern $[2^-(02^-)^{*r}0$ can overlap with an occurrence of a pattern $\{1^+,[\}1\{1^+,]\}$, or $1^+2^-(02^-)^{+r}1^+$,*
3. *no occurrence of the pattern $02^-(02^-)^{*r}]$ can overlap with an occurrence of a pattern $\{1^+,[\}1\{1^+,]\}$ or $1^+2^-(02^-)^{+r}1^+$,*

4. *if two occurrences of the patterns* $[2^-(02^-)^{*r}0$ *and* $02^-(02^-)^{*r}]$ *overlap, then* CT *is a caterpillar with pattern* $[2^-(02^-)^{*r}]$.

We first prove that every caterpillar with no pair of adjacent trunks admits an independent broadcast f with $\text{cost}(f) = \beta^*(CT)$.

Lemma 10 *Every caterpillar* $CT = CT(\lambda_0, \dots, \lambda_k)$ *of length* $k \geq 1$*, with no pair of adjacent trunks, admits an independent broadcast* f *with* $\text{cost}(f) = \beta^*(CT)$.

Proof. We will construct a sequence of independent broadcasts $f_1, \dots, f_4$, step by step, such that $\text{cost}(f_4) = \beta^*(CT)$. Each independent broadcast f_i, $2 \leq i \leq 4$, is obtained by possibly modifying the independent broadcast f_{i-1} and is such that $\text{cost}(f_i) \geq \text{cost}(f_{i-1})$. Moreover, for each independent broadcast f_i, $1 \leq i \leq 4$, we will have $f_i(v) = 0$ whenever v is a stem. These modifications are illustrated on Figures 3 and 4, using the same drawing conventions as in Figure 2. Only useful broadcast values are given in these figures. These figures should help the reader to see that all the proposed modifications lead to a new valid independent broadcast.

Step 1. Let f_1 be the mapping defined by $f_1(v) = 1$ if v is a pendent vertex or a trunk, and $f_1(v) = 0$ otherwise. Clearly, f_1 is an independent broadcast on CT with

$$\text{cost}(f_1) = \lambda(CT) + \tau(CT).$$

Step 2. Let f_2 be the mapping defined by $f_2(v) = 2$ if $v = \ell_i^1$ for some i, $0 \leq i \leq k$, such that (i) $\lambda_i = 1$, (ii) $i = 0$ or $\lambda_{i-1} \geq 1$, and (iii) $i = k$ or $\lambda_{i+1} \geq 1$, and $f_2(v) = f_1(v)$ otherwise (see Figure 3(a)). Again, f_2 is an independent broadcast on CT with

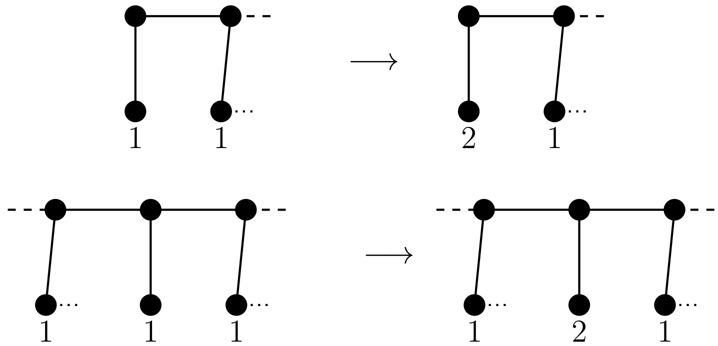
$$\text{cost}(f_2) = \text{cost}(f_1) + \#_{CT}(\{1^+, [\}1\{1^+,]\}).$$

Step 3. Suppose that CT contains the pattern $1^+2^-(02^-)^{+r}1^+$, of length $2r + 3$, and let $M = CT[i_0, i_0 + 2r + 2]$ be the corresponding occurrence of this pattern. We thus have $f_2(v) = 1$ for every trunk of M and for every pendent neighbor of a stem vertex v_j on M with $i_0 + 1 \leq j \leq i_0 + 2r + 1$. Hence, the cost of the restriction f_2' of f_2 to M is

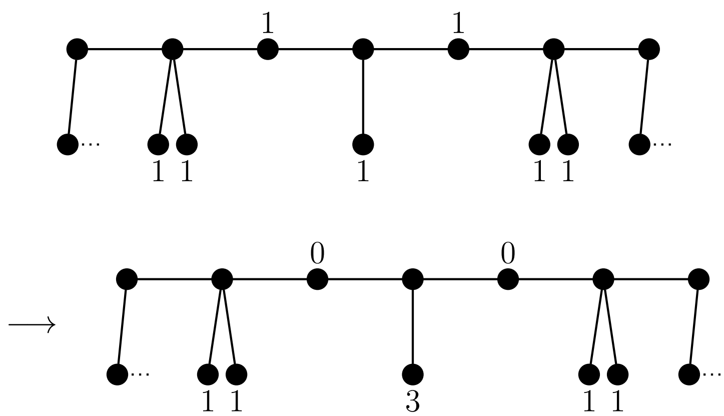
$$\text{cost}(f_2') = f_2^*(v_{i_0}) + \lambda(M[i_0 + 1, i_0 + 2r + 1]) + \tau(M) + f_2^*(v_{i_0+2r+2}).$$

Let f_3 be the mapping first defined by $f_3(v) = f_2(v)$ for every vertex v. We then modify f_3 as follows. If the subgraph $M[i_0+1, i_0+2r+1]$ contains no stem vertex v_i with $\lambda_i = 1$, we keep $f_3 = f_2$. Otherwise, we let

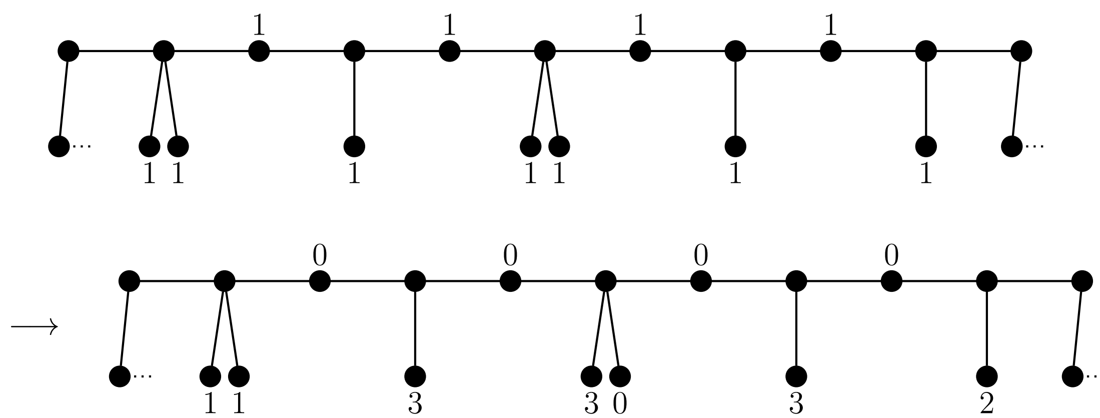
- $f_3(\ell_{i_0+1}^1) = 2$ if $\lambda_{i_0+1} = 1$,
- $f_3(\ell_{i_0+2r+1}^1) = 2$ if $\lambda_{i_0+2r+1} = 1$,



(a) From f_1 to f_2



(b) From f_2 to f_3, pattern $1^{+}201021^{+}$, $\mathrm{cost}(f_3') = \mathrm{cost}(f_2') + (1-1)$



(c) From f_2 to f_3, pattern $1^{+}2010201011^{+}$, $\mathrm{cost}(f_3') = \mathrm{cost}(f_2') + (3-1)$

Figure 3: Proof of Lemma 10: from f_1 to f_3

- $f_3(\ell^1_{i_0+2j+1}) = 3$ (and $f_3(\ell^2_{i_0+2j+1}) = 0$ if $\lambda_{i_0+2j+1} = 2$) for every j, $1 \leq j \leq r-1$,
- $f_3(v_{i_0+2j}) = 0$ for every j, $1 \leq j \leq r$,

(see Figure 3(b) and (c)). The cost of the restriction f'_3 of f_3 on M is then

$$\text{cost}(f'_3) = \text{cost}(f'_2) + \max\{0, \#_{M[i_0+1,i_0+2r+1]}(1) - 1\} = \text{cost}(f'_2) + \alpha_1(M).$$

By Observation 9, two occurrences of the pattern $1^+2^-(02^-)^{+r}1^+$ can only overlap on their end-vertices. Therefore, doing the above modification for every occurrence of the pattern $1^+2^-(02^-)^{+r}1^+$ in M, the so-obtained independent broadcast f_3 satisfies

$$\text{cost}(f_3) = \text{cost}(f_2) + \alpha_1(CT).$$

Step 4. Suppose first that CT contains the pattern $02^-(02^-)^{*r}0$, of length $2r+3$, and let $M = CT[i_0, i_0+2r+2]$, $i_0 \geq 1$, $i_0+2r+2 \leq k-1$, be the corresponding occurrence of this pattern. We thus have $f_2(v) = 1$ for every trunk of M and for every pendent neighbor of a stem vertex v_j on M with $i_0+1 \leq j \leq i_0+2r+1$. Hence, the cost of the restriction f'_3 of f_3 to M is

$$\text{cost}(f'_3) = f^*_3(v_{i_0}) + \lambda(M) + \tau(M[i_0+1, i_0+2r+1]) + f^*_3(v_{i_0+2r+2}).$$

Let f_4 be the mapping first defined by $f_4(v) = f_3(v)$ for every vertex v. We then modify f_4 as follows. If the subgraph $M[i_0+1, i_0+2r+1]$ contains no stem vertex v_i with $\lambda_i = 1$, we keep $f_4 = f_3$. Otherwise, we let

- $f_4(\ell^1_{i_0+2j+1}) = 3$ (and $f_4(\ell^2_{i_0+2j+1}) = 0$ if $\lambda_{i_0+2j+1} = 2$) for every j, $0 \leq j \leq r$,
- $f_4(v_{i_0+2j}) = 0$ for every j, $0 \leq j \leq r$,

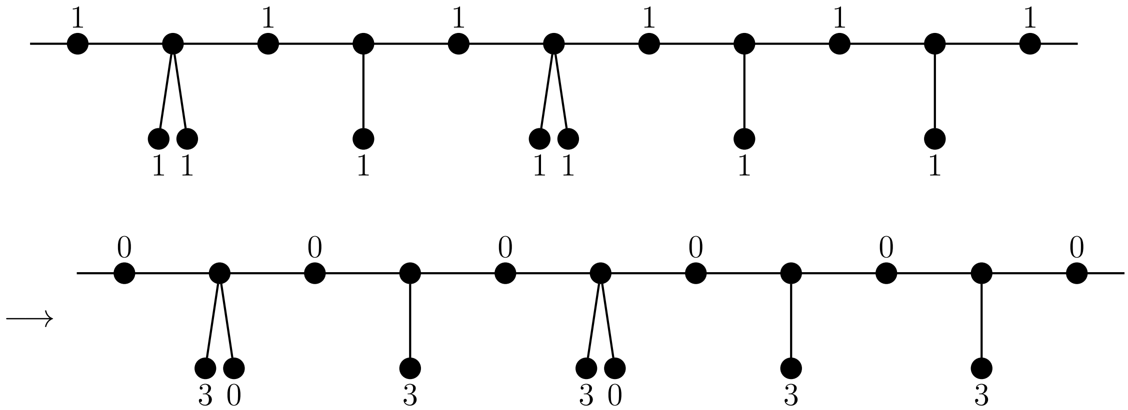
(see Figure 4(a)). The cost of the restriction f'_4 of f_4 on M is then

$$\text{cost}(f'_4) = \text{cost}(f'_3) + \max\{0, \#_M(1) - 1\} = \text{cost}(f'_3) + \alpha_2(M).$$

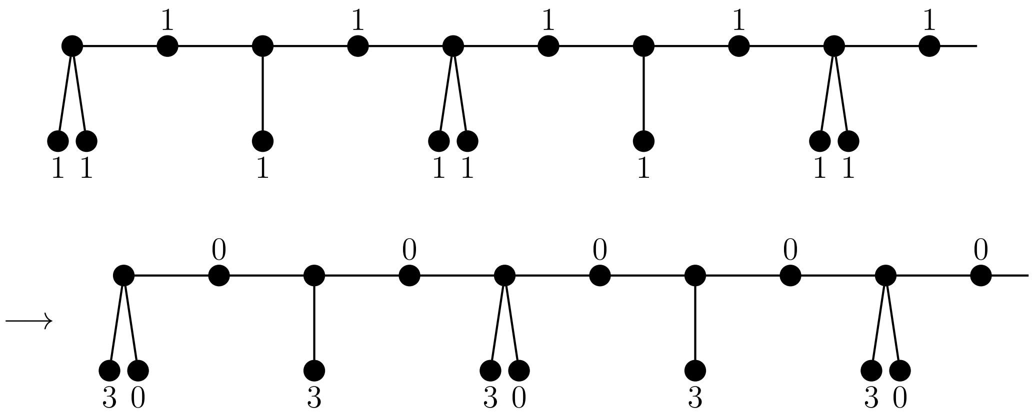
Suppose now that CT contains the pattern $[2^-(02^-)^{*r}0$, of length $2r+2$, and let $M = CT[0, 2r+1]$ be the corresponding occurrence of this pattern. Doing the same type of modification as above (see Figure 4(b)), the cost of the restriction f'_4 of f_4 on M is then

$$\text{cost}(f'_4) = \text{cost}(f'_3) + \max\{0, \#_M(1) - 1\} + 1 = \text{cost}(f'_3) + \alpha_2(M).$$

Finally, if CT contains the pattern $02^-(02^-)^{*r}]$ and CT is not a caterpillar with pattern $[2^-(02^-)^{*r}]$, the same type of modification leads to the same property.



(a) From f_3 to f_4, pattern 02010201010, $\mathrm{cost}(f'_4) = \mathrm{cost}(f'_3) + (3-1) + 0$



(b) From f_3 to f_4, pattern [2010201020, $\mathrm{cost}(f'_4) = \mathrm{cost}(f'_3) + (2-1) + 1$

Figure 4: Proof of Lemma 10: from f_3 to f_4

By Observation 9, no two occurrences of the patterns $02^-(02^-)^{*r}0$ and $[2^-(02^-)^{*r}0$ (or $02^-(02^-)^{*r}0$ and $02^-(02^-)^{*r}]$) can overlap. Therefore, doing the above modification for every occurrence of these patterns in M, the so-obtained independent broadcast f_4 satisfies

$$\mathrm{cost}(f_4) = \mathrm{cost}(f_3) + \alpha_2(CT) = \beta^*(CT).$$

This completes the proof. □

The next lemma shows that if f is an optimal non-canonical independent broadcast on a caterpillar CT with no pair of adjacent trunks, with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$, then there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT such that the $\tilde{f}$-values of the pendent neighbors of v_0 and v_k only depend on the values of λ_0, λ_1 and λ_{k-1}, λ_k, respectively:

Lemma 11 *Let $CT = CT(\lambda_0, \dots, \lambda_k)$ be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks. If f is an optimal non-canonical independent broadcast on CT with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT)-1)$, then there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT, thus with $\mathrm{cost}(\tilde{f}) = \mathrm{cost}(f)$, such that, for every $i \in \{0, k\}$, we have*

1. *if $\lambda_i = 1$ and $\lambda_{i'} \geq 1$, then $\tilde{f}(\ell_i^1) = 2$,*
2. *if $\lambda_i = 1$ and $\lambda_{i'} = 0$, then $\tilde{f}(\ell_i^1) = 3$,*
3. *if $\lambda_i = 2$ and $\lambda_{i'} \geq 1$, then $\tilde{f}(\ell_i^1) = \tilde{f}(\ell_i^2) = 1$,*
4. *if $\lambda_i = 2$ and $\lambda_{i'} = 0$, then $\tilde{f}(\ell_i^1) = 3$ and $\tilde{f}(\ell_i^2) = 0$,*
5. *if $\lambda_i \geq 3$, then $\tilde{f}(\ell_i^j) = 1$ for every j, $1 \leq j \leq \lambda_i$,*

where $i' = 1$ if $i = 0$, or $i' = k - 1$ if $i = k$.

Proof. Note first that if such a broadcast $\tilde{f}$ exists, then, by Lemma 6, $\tilde{f}(u) = 0$ for every stem u of CT. Therefore, the value of $\sum_{1 \leq j \leq \lambda_i} \tilde{f}(\ell_i^j)$ cannot be strictly less than the value claimed in the lemma since otherwise it would contradict the optimality of $\tilde{f}$.

By symmetry, it is enough to prove the lemma for the pendent neighbors of v_0. Let $CT_0 = CT(\lambda_0, \dots, \lambda_k)$ be a minimal counterexample, with respect to the subgraph order, to the lemma. That is, every sub-caterpillar of CT_0 satisfies the statement of the lemma and, for every optimal non-canonical independent broadcast f on CT_0 with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$, there is a pendent neighbor, say ℓ_0^1 without loss of generality, of v_0 such that $f(\ell_0^1) = x$ and x is strictly greater than the value claimed by the lemma (note that, in case 3, if $f(\ell_0^1) = 2$ (resp. 0) and $f(\ell_0^2) = 0$ (resp. 2), then we can equivalently assign the value 1 to both of them). We will prove that such a minimal counterexample cannot exist.

Let f_0 be any such independent broadcast on CT_0 for which the value $f(\ell_0^1) = x$ is minimal. We thus have $x \geq 3$ whenever $\lambda_1 > 0$ or $\lambda_0 \geq 3$ (since in this latter case we can assign value 1 to each of the at least three pendent neighbors of v_0, and thus $x = 2$ would imply that f_0 is not optimal), and $x \geq 4$ whenever $\lambda_1 = 0$.

Since $f_0(\ell_0^1) = x > 1$, we have $f_0^*(v_i) = 0$ for every i, $1 \leq i \leq x - 2$, and $f_0(v_{x-1}) = 0$. Moreover, $x - 1 < k$ since f_0 is a non-canonical independent broadcast, and v_{x-1} cannot be a trunk, since otherwise we could set $f_0(\ell_0^1) = x+1$ (recall that, by Lemma 6, $f_0(v_i) = 0$ for every stem v_i, and thus $f_0(v_x) = 0$), contradicting the optimality of f_0.

Let now $CT_1 = (\lambda_{x-1}, \dots, \lambda_k)$ be the caterpillar obtained from CT_0 by deleting vertices $v_0, \dots, v_{x-2}$ and their pendent neighbors (see Figure 5(a)). Note that $f_0(u) = 0$ for every such deleted vertex $u \neq \ell_0^1$. Let f_1 denote the restriction of f_0 to $V(CT_1)$. Since $f_0(\ell_0^1) = x$, we get

$$f_1(u) = f_0(u) \leq \max\{e_{CT_1}(u), d_{CT_0}(u, \ell_0^1)\} \leq e_{CT_1}(u)$$

for every vertex $u \in V(CT_1)$, so that f_1 is an independent broadcast on CT_1 by Observation 4. Moreover, since $\mathrm{diam}(CT_1) = \mathrm{diam}(CT_0) - x + 1$, we have

$$\mathrm{cost}(f_1) = \mathrm{cost}(f_0) - x > 2(\mathrm{diam}(CT_0) - 1) - x = 2(\mathrm{diam}(CT_1) - 1) + x - 2.$$

Since $x > 1$, we thus have $\mathrm{cost}(f_1) \geq 2(\mathrm{diam}(CT_1) - 1)$. Therefore, since CT_0 is a minimal counterexample, we get that either f_1 is a canonical independent broadcast on CT_1 or there exists an optimal non-canonical independent broadcast f'_1 on CT_1 with $\mathrm{cost}(f'_1) \geq \mathrm{cost}(f_1)$ and f'_1 satisfies the statement of the lemma.

Suppose first that f_1 is a canonical independent broadcast. This implies

$$\mathrm{cost}(f_1) = 2(\mathrm{diam}(CT_1) - 1).$$

Hence,

$$\mathrm{cost}(f_0) = \mathrm{cost}(f_1) + x = 2(\mathrm{diam}(CT_1) - 1) + x < 2(\mathrm{diam}(CT_0) - 1),$$

which contradicts our assumption on $\mathrm{cost}(f_0)$.

Therefore, there exists an optimal non-canonical independent broadcast f'_1 on CT_1 with $\mathrm{cost}(f'_1) \geq \mathrm{cost}(f_1)$ satisfying the statement of the lemma. If $\mathrm{cost}(f'_1) > \mathrm{cost}(f_1)$, the mapping f'_0 given by $f'_0(u) = f'_1(u)$ for every vertex $u \in V(CT_1)$ and $f'_0(u) = f_0(u)$ for every vertex $u \in V(CT_0) \setminus V(CT_1)$, is a non-canonical independent broadcast f'_0 on CT_0 (since $x \geq 3$) that contradicts the optimality of f_0.

Hence, f_1 is optimal and thus satisfies the statement of the lemma. Let $\tilde{f}_1$ be the non-canonical independent broadcast satisfying items 1 to 5 of the lemma, and let

$$m = \max\left\{\tilde{f}_1(\ell^j_{x-1}),\ 1 \leq j \leq \lambda_{x-1}\right\}.$$

We consider two cases, depending on whether v_{x-2} is a stem or not. Recall that $v_{x-2} \neq v_0$, since $x \geq 3$.

1. $\lambda_{x-2} > 0$.
   Let f'_0 be the non-canonical independent broadcast on CT_0 given by $f'_0(\ell^1_0) = x - 1$, $f'_0(\ell^1_{x-2}) = 2$, $f'_0(u) = 0$ for every vertex $u \in V(CT_0) \setminus (V(CT_1) \cup \{\ell^1_0, \ell^1_{x-2}\})$, and either $f'_0(u) = \tilde{f}_1(u)$ for every vertex $u \in V(CT_1)$, if $m \leq 2$ (see Figure 5(b)), or $f'_0(\ell^1_{x-1}) = 2$ and $f'_0(u) = \tilde{f}_1(u)$ for every vertex $u \in V(CT_1) \setminus \{\ell^1_{x-1}\}$, if $m = 3$ (see Figure 5(c)). We then get $\mathrm{cost}(f'_0) = \mathrm{cost}(f_0) + 1$ if $m \leq 2$, contradicting the optimality of f_0, or $\mathrm{cost}(f'_0) = \mathrm{cost}(f_0)$ if $m = 3$, in which case either f'_0 satisfies items 1 to 5 of the lemma or contradicts the minimality of x.

2. $\lambda_{x-2} = 0$.
   If $x = 3$, then $\lambda_1 = 0$ which implies $x \geq 4$, a contradiction. Hence, we have $x \geq 4$, and thus $v_{x-3} \neq v_0$. Let f'_0 be the non-canonical independent broadcast on CT_0 given by $f'_0(\ell^1_0) = x - 2$, $f'_0(\ell^1_{x-3}) = 2$, $f'_0(u) = 0$ for every vertex $u \in V(CT_0) \setminus (V(CT_1) \cup \{\ell^1_0, \ell^1_{x-3}\})$, and $f'_0(u) = \tilde{f}_1(u)$ for every vertex $u \in V(CT_1)$ (see Figure 5(d)). We then get $\mathrm{cost}(f'_0) = \mathrm{cost}(f_0)$, and thus either f'_0 satisfies items 1 to 5 of the lemma or contradicts the minimality of x.

This concludes the proof. □

We now consider the internal stems of a caterpillar. The next lemma shows that if f is an optimal non-canonical independent broadcast on a caterpillar CT with no pair of adjacent trunks, with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$, then there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT such that $\tilde{f}^*(v_i) > 0$ for every internal stem v_i of CT, $1 \leq i \leq k-1$.

Lemma 12 *Let $CT = CT(\lambda_0, \dots, \lambda_k)$ be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks. If f is an optimal non-canonical independent broadcast on CT with* $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$*, then there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT, thus with* $\mathrm{cost}(\tilde{f}) = \mathrm{cost}(f)$*, such that:*

1. *$\tilde{f}$ satisfies the five items of Lemma 11,*
2. *for every i, $1 \leq i \leq k-1$, if $\lambda_i > 0$, then $\tilde{f}^*(v_i) > 0$.*

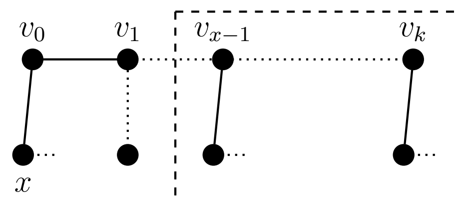
Proof. We know by Lemma 11 that there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT, with $\mathrm{cost}(\tilde{f}) = \mathrm{cost}(f)$, satisfying the five items of Lemma 11. Moreover, one suppose that $\tilde{f}$ has been chosen in such a way that $V^+_{\tilde{f}}$ contains the largest possible number of pendent vertices.

Suppose to the contrary that there exists a vertex v_i, $1 \leq i \leq k-1$, with $\lambda_i > 0$ and $\tilde{f}^*(v_i) = 0$, and that for every $j < i$, $\tilde{f}^*(v_j) > 0$ whenever $\lambda_j > 0$. We consider three cases.

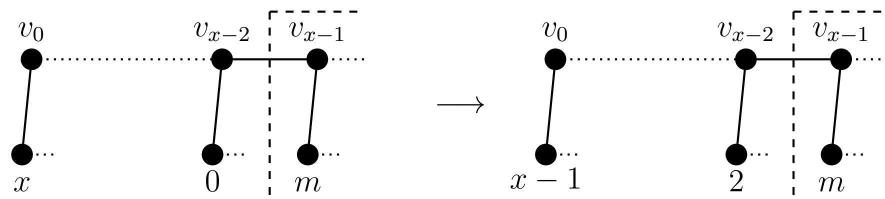
1. $i = 1$ or $i = k-1$.
   By symmetry, it suffices to consider the case $i = 1$. By Lemma 11, we know that $\tilde{f}(\ell_0^j) \leq 2$ for every j, $1 \leq j \leq \lambda_0$. Therefore, no pendent neighbor of v_1 is $\tilde{f}$-dominated by a pendent neighbor of v_0. Let y be the vertex of CT that $\tilde{f}$-dominates the pendent neighbors of v_1 (note that y is necessarily unique), and g be the mapping defined as follows. For every vertex u of CT, let

$$g(u) = \begin{cases} \tilde{f}(y) - 1 & \text{if } u = y, \\ 1 & \text{if } u = \ell_1^1, \\ 1 & \text{if } u \neq \ell_1^1,\ u \text{ is } \tilde{f}\text{-dominated only by } y \text{ and } d_{CT}(u, y) = \tilde{f}(y), \\ \tilde{f}(u) & \text{otherwise.} \end{cases}$$

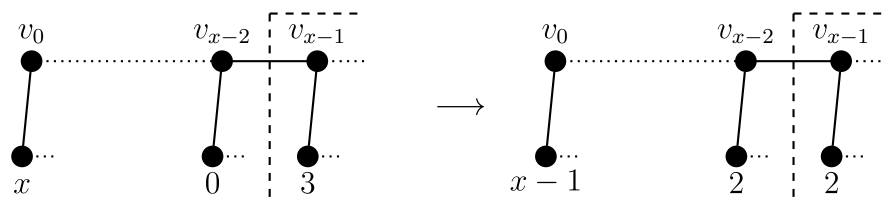
   We claim that the mapping g is a non-canonical independent broadcast on CT with $\mathrm{cost}(g) \geq \mathrm{cost}(\tilde{f})$. Indeed, all vertices x with $d_{CT}(x, y) < \tilde{f}(y)$ that were $\tilde{f}$-dominated by y are still g-dominated by y, and all vertices $x' \neq \ell_1^1$ with $d_{CT}(x', y) = \tilde{f}(y)$ that were $\tilde{f}$-dominated only by y are now g-broadcast vertices with $g(x') = 1$



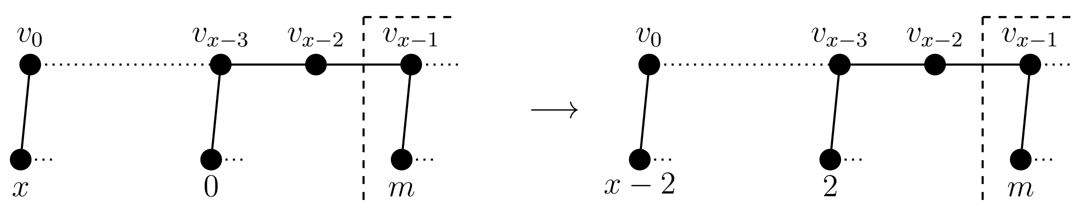
(a) The sub-caterpillar CT_1



(b) $\lambda_{x-2} > 0$ and $m \leq 2$



(c) $\lambda_{x-2} > 0$ and $m = 3$



(d) $\lambda_{x-2} = 0$

Figure 5: Configurations for the proof of Lemma 11

(note that since every such x' was $\tilde{f}$-dominated only by y, we have $g(z) = \tilde{f}(z) = 0$ for every neighbor z of x').

Now, if there exists a vertex z which is $\tilde{f}$-dominated only by y, we get $\mathrm{cost}(g) \geq \mathrm{cost}(\tilde{f}) + 1$, contradicting the optimality of $\tilde{f}$. If no such vertex exists, we get $\mathrm{cost}(g) = \mathrm{cost}(\tilde{f})$ and V_g^+ contains more pendent vertices than $V_{\tilde{f}}^+$, contrary to our assumption.

2. $i = 2$ and $\lambda_1 = 0$, or $i = k-2$ and $\lambda_{k-1} = 0$.
By symmetry, it suffices to consider the case $i = 2$. By Lemma 11, we know that $\tilde{f}(\ell_0^j) \leq 3$ for every j, $1 \leq j \leq \lambda_0$. Therefore, no pendent neighbor of v_2 is $\tilde{f}$-dominated by a pendent neighbor of v_0. Let y be the (unique) vertex of CT that $\tilde{f}$-dominates the pendent neighbors of v_2 (note that we necessarily have $\tilde{f}(y) \geq 2$).

If $y = v_3$ and $\tilde{f}(v_3) = 3$ (since $\tilde{f}^*(v_0) > 0$, we necessarily have $\tilde{f}(v_3) \leq 3$), we define the mapping g as follows. For every vertex u of CT, let

$$g(u) = \begin{cases} 0 & \text{if } u = v_3, \\ 3 & \text{if } u = \ell_2^1, \\ 1 & \text{if } u \neq \ell_2^1,\ u \text{ is } \tilde{f}\text{-dominated only by } v_3 \text{ and } d_{CT}(u,y) = 2, \\ \tilde{f}(u) & \text{otherwise.} \end{cases}$$

Otherwise (including the case $y = v_3$ and $\tilde{f}(v_3) = 2$), the mapping g is defined by

$$g(u) = \begin{cases} \tilde{f}(y) - 2 & \text{if } u = y, \\ 2 & \text{if } u = \ell_2^1, \\ 1 & \text{if } u \neq \ell_2^1,\ u \text{ is } \tilde{f}\text{-dominated only by } y \text{ and } d_{CT}(u,y) = \tilde{f}(y) - 1, \\ \tilde{f}(u) & \text{otherwise,} \end{cases}$$

for every vertex u of CT.

In both cases, the mapping g is again a non-canonical independent broadcast on CT with $\mathrm{cost}(g) \geq \mathrm{cost}(\tilde{f})$. Indeed, all vertices x with $d_{CT}(x,y) < \tilde{f}(y) - 1$ that were $\tilde{f}$-dominated by y are g-dominated by ℓ_1^2 (if $y = v_3$) or still g-dominated by y (if $y \neq v_3$), and all vertices $x' \neq \ell_2^1$ with $\tilde{f}(y) - 1 \leq d_{CT}(x', y) \leq \tilde{f}(y)$ that were $\tilde{f}$-dominated only by y are now either g-broadcast vertices (if $d_{CT}(x', y) = \tilde{f}(y) - 1$) or g-dominated by a vertex x'' with $d_{CT}(x'', y) = \tilde{f}(y) - 1$ and $g(x'') = 1$.

We then get a contradiction as in Case 1.

3. $2 < i < k-2$, or $i = 2$ and $\lambda_1 > 0$, or $i = k-2$ and $\lambda_{k-1} > 0$.
In that case, we have $\tilde{f}^*(v_j) > 0$ for every vertex v_j with $j < i$ and $\lambda_j > 0$. Note also that we have at least two such vertices v_j with $j < i$ and $\lambda_j > 0$.

By symmetry, it suffices to consider the cases $2 < i < k-2$, and $i = 2$ (with $\lambda_1 > 0$). We consider three subcases.

(a) Suppose first that the pendent neighbors of v_i are $\tilde{f}$-dominated only by a vertex $y = v_{j_0}$ or $y = \ell_{j_0}^{k_0}$ with $j_0 < i$ and $1 \leq k_0 \leq \lambda_{j_0}$. Observe that the pendent neighbors of v_i cannot be $\tilde{f}$-dominated by two such vertices, say y and y', since we would have $d_{CT}(y, y') < d_{CT}(y, \ell_i^1)$ so that $\tilde{f}$ would not be independent.

Since $\tilde{f}^*(v_j) > 0$ for every $j < i$ such that $\lambda_j > 0$, we necessarily have, by Lemma 6, either y is a pendent neighbor of v_{i-1}, if $\lambda_{i-1} > 1$, or a pendent neighbor of v_{i-2}, if $\lambda_{i-1} = 0$. Moreover, since $\tilde{f}^*(v_j) > 0$ for every $j < i$ such that $\lambda_j > 0$, and since we have at least two such vertices, we necessarily have $\tilde{f}(y) \leq 3$. This implies in particular $\lambda_{i-1} > 0$, as otherwise we would have $\tilde{f}(y) \leq 3$ and $d_{CT}(y, \ell_i^1) = 4$, contradicting the fact that y $\tilde{f}$-dominates ℓ_i^1, and thus y is a pendent neighbor of v_{i-1}.

Let now g be the mapping defined as follows. For every vertex u of CT, let

$$g(u) = \begin{cases} \tilde{f}(y) - 1 & \text{if } u = y, \\ 1 & \text{if } u = \ell_i^1, \\ 1 & \text{if } u \neq \ell_i^1,\ u \text{ is } \tilde{f}\text{-dominated only by } y \text{ and } d_{CT}(u, y) = \tilde{f}(y), \\ \tilde{f}(u) & \text{otherwise.} \end{cases}$$

Again, the mapping g is a non-canonical independent broadcast on CT with $\text{cost}(g) \geq \text{cost}(\tilde{f})$. Indeed, all vertices x with $d_{CT}(x, y) < \tilde{f}(y)$ that were $\tilde{f}$-dominated by y are still g-dominated either by y, and all vertices $x' \neq \ell_i^1$ with $d_{CT}(x', y) = \tilde{f}(y)$ that were $\tilde{f}$-dominated only by y are now g-broadcast vertices.

We then get a contradiction as in Cases 1 and 2.

(b) Suppose now that the pendent neighbors of v_i are $\tilde{f}$-dominated only by a vertex $y = v_{j_0}$ (with $\lambda_{j_0} = 0$) or $y = \ell_{j_0}^{k_0}$ ($1 \leq k_0 \leq \lambda_{j_0}$), with $j_0 > i$. Observe that, using the same argument as in Case (a), such a vertex y must be unique.

Moreover, we necessarily have $\tilde{f}(y) \geq 2$.

If $\lambda_{i-1} = 0$, we consider two cases, as we did in Case 2. If $y = v_{i+1}$ and $\tilde{f}(v_{i+1}) = 3$, we define the mapping g by

$$g(u) = \begin{cases} 0 & \text{if } u = v_{i+1}, \\ 3 & \text{if } u = \ell_i^1, \\ 1 & \text{if } u \neq \ell_i^1,\ u \text{ is } \tilde{f}\text{-dominated only by } y \text{ and } d_{CT}(u, y) = 2, \\ \tilde{f}(u) & \text{otherwise,} \end{cases}$$

for every vertex u of CT. Otherwise, the mapping g is defined by

$$g(u) = \begin{cases} \tilde{f}(y) - 2 & \text{if } u = y, \\ 2 & \text{if } u = \ell_i^1, \\ 1 & \text{if } u \neq \ell_i^1,\ u \text{ is } \tilde{f}\text{-dominated only by } y \text{ and } d_{CT}(u, y) = \tilde{f}(y) - 1, \\ \tilde{f}(u) & \text{otherwise,} \end{cases}$$

for every vertex u of CT.

Otherwise, that is, $\lambda_{i-1} > 0$, we define the mapping g as follows. For every vertex u of CT, let

$$g(u) = \begin{cases} \tilde{f}(y) - 1 & \text{if } u = y, \\ 1 & \text{if } u = \ell_i^1, \\ 1 & \text{if } u \neq \ell_i^1,\ u \text{ is } \tilde{f}\text{-dominated only by } y \text{ and } d_{CT}(u, y) = \tilde{f}(y), \\ \tilde{f}(u) & \text{otherwise.} \end{cases}$$

Again, using similar arguments, in each case the above-defined mapping is a non-canonical independent broadcast on CT with $\mathrm{cost}(g) \geq \mathrm{cost}(\tilde{f})$ and the contradiction arises as in Cases 1 and 2.

(c) Suppose finally that the pendent neighbors of v_i are $\tilde{f}$-dominated both by a vertex $y_1 = v_{j_1}$ or $y_1 = \ell_{j_1}^{k_1}$ with $j_1 < i$ and $1 \leq k_1 \leq \lambda_{j_1}$, and by a vertex $y_2 = v_{j_2}$ or $y_2 = \ell_{j_2}^{k_2}$ with $j_2 > i$ and $1 \leq k_2 \leq \lambda_{j_2}$ (again, both y_1 and y_2 must be unique). In that case, as discussed in Case (a) above, we necessarily have $\lambda_{i-1} > 0$. Moreover, we necessarily have $\tilde{f}(y_1) = 3$ and $\tilde{f}(y_2) \geq 2$.

Let now g be the mapping defined as follows. For every vertex u of CT, let

$$g(u) = \begin{cases} \tilde{f}(y_1) - 1 & \text{if } u = y_1, \\ \tilde{f}(y_2) - 1 & \text{if } u = y_2, \\ 2 & \text{if } u = \ell_i^1, \\ 1 & \text{if } u \neq \ell_i^1,\ u \text{ is } \tilde{f}\text{-dominated only by } y_2 \text{ and } d_{CT}(u, y_2) = \tilde{f}(y_2), \\ \tilde{f}(u) & \text{otherwise.} \end{cases}$$

Note here that no vertex at distance $\tilde{f}(y_1)$ from y_1 can be $\tilde{f}$-dominated only by y_1. Indeed, suppose that such a vertex, say w, exists. Clearly, w cannot be "to the left of v_i" since this would imply $w = v_{i-3}$ and $\lambda_{i-2} = 0$, but in that case w is also $\tilde{f}$-dominated by at least one of its pendent neighbors. On the other hand, w cannot be "to the right of v_i" since in that case w would also be $\tilde{f}$-dominated by y_2.

Again, using similar arguments, the above-defined mapping is a non-canonical independent broadcast on CT with $\mathrm{cost}(g) \geq \mathrm{cost}(\tilde{f})$ and the contradiction arises as in Cases 1 and 2.

We thus get a contradiction in each case. This completes the proof. □

Our aim now is to prove that if f is an optimal non-canonical independent broadcast on a caterpillar CT with no pair of adjacent trunks, with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$, then $\mathrm{cost}(f) = \mathrm{cost}(\beta^*)$ (Lemma 16 below). We first prove that for every such broadcast f, $f(v_i) \leq 1$ for every trunk v_i. This easily follows from Lemma 12.

Lemma 13 *Let $CT = CT(\lambda_0, \ldots, \lambda_k)$ be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks. If f is an optimal non-canonical independent broadcast on CT with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT)-1)$, then there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT, thus with $\mathrm{cost}(\tilde{f}) = \mathrm{cost}(f)$, such that:*

1. *$\tilde{f}$ satisfies the two items of Lemma 12,*
2. *for every i, $1 \leq i \leq k-1$, if $\lambda_i = 0$, then $\tilde{f}^*(v_i) \leq 1$.*

Proof. We know by Lemma 12 that there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT satisfying the two items of Lemma 12, so that, in particular, $\tilde{f}^*(v_j) > 0$ for every stem v_j, $0 \leq j \leq k$. Since CT has no pair of adjacent trunks, and $\tilde{f}$ is independent, we thus necessarily have $\tilde{f}^*(v_i) \leq 1$ for every trunk v_i, $1 \leq i \leq k-1$. □

Finally, the next lemma will show that the cost of any optimal non-canonical independent broadcast on a caterpillar CT of length $k \geq 1$ with no pair of adjacent trunks cannot exceed the value $\beta^*(CT)$.

We first introduce a few more notation. Let CT be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks. We denote by σ a sequence of ℓ consecutive spine vertices in CT, that is, $\sigma = v_i \dots v_{i+\ell-1}$, with $\ell \leq k+1$ and $0 \leq i \leq k-\ell+1$. For such a given sequence $\sigma = v_i \dots v_{i+\ell-1}$, we denote by t_σ the number of trunks in σ, that is,

$$t_\sigma = |\{v_j \mid i \leq j \leq i+\ell-1 \text{ and } \lambda_j = 0\}|\,.$$

If f is an independent broadcast on CT, we then denote by $f^*(\sigma)$ the *weight of* σ, that is,

$$f^*(\sigma) = \sum_{0 \leq j \leq \ell-1} f^*(v_{i+j}).$$

Lemma 14 *Let $CT = CT(\lambda_0, \dots, \lambda_k)$ be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks, and f be an optimal non-canonical independent broadcast on CT with* $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$*. Then there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT, thus with* $\mathrm{cost}(\tilde{f}) = \mathrm{cost}(f)$*, such that:*

1. *$\tilde{f}$ satisfies the two items of Lemma 13.*
2. *For every i, $0 \leq i \leq k$, if $\lambda_i \geq 3$, then $\tilde{f}^*(v_i) \leq \lambda_i$.*
3. *If $v_a v_{a+1}$, $0 \leq a < k$, is an occurrence of the pattern 1^+2^- (resp. of the pattern 2^-1^+), then $\tilde{f}^*(v_{a+1}) \leq 2$ (resp. $\tilde{f}^*(v_a) \leq 2$).*
4. *If $v_a \sigma v_b$ is an occurrence of the pattern $1^+2^-(02^-)^{+r}1^+$, then $\tilde{f}^*(\sigma) \leq 3t_\sigma + 2$ if $v_a \sigma v_b$ is an occurrence of the pattern $1^+2(02)^{+r}1^+$, and $\tilde{f}^*(\sigma) \leq 3t_\sigma + 1$ otherwise.*
5. *If σ is an occurrence of the pattern $02^-(02^-)^{*r}0$, then $\tilde{f}^*(\sigma) \leq 3t_\sigma - 2$ if $v_a \sigma v_b$ is an occurrence of the pattern $02(02)^{*r}0$, and $\tilde{f}^*(\sigma) \leq 3t_\sigma - 3$ otherwise.*
6. *If σ is an occurrence of the pattern $[2^-(02^-)^{*r}0$ or of the pattern $02^-(02^-)^{*r}]$, then $\tilde{f}^*(\sigma) \leq 3t_\sigma$.*

Proof. We consider the six items of the lemma.

1. We know by Lemma 13 that there exists an optimal non-canonical independent broadcast $\tilde{f}$ on CT satisfying the two items of Lemma 13, so that, in particular, $\tilde{f}^*(v_i) > 0$ for every stem v_i, $0 \le i \le k$ and $\tilde{f}^*(v_j) \le 1$ for every trunk v_j, $1 \le j \le k-1$. We thus assume for all following items that such an optimal non-canonical independent broadcast $\tilde{f}$ on CT has been chosen.

2. Suppose to the contrary that there exists some i, $0 \le i \le k$, with $\tilde{f}^*(v_i) > \lambda_i \ge 3$. This implies that v_i has exactly one pendent neighbor, say ℓ_i^1 without loss of generality, which is an $\tilde{f}$-broadcast vertex. Since $\tilde{f}(\ell_i^1) \ge 4$, we necessarily have a stem v with $d_{CT}(v_i, v) \le 2$ and $\tilde{f}^*(v) = 0$, contradicting our assumption that $\tilde{f}$ satisfies Lemma 12.

3. Let $v_a v_{a+1}$, $0 \le a < k$, be an occurrence of the pattern 1^+2^- (the case 2^-1^+ is similar, by symmetry). By Lemmas 6 and 12, we now that $\tilde{f}^*(v_a) > 0$ and $\tilde{f}(v_a) = 0$. This clearly implies $\tilde{f}^*(v_{a+1}) \le 2$.

4. Let $v_a \sigma v_b = v_i v_{i+1} \dots v_{i+2r+2}$ be an occurrence of the pattern $1^+2(02)^{+r}1^+$, for some i, $0 \le i \le k-2r-2$. We thus have $t_\sigma = r$. Since $\tilde{f}$ satisfies Lemma 13, we have $\tilde{f}^*(v_i) > 0$, $\tilde{f}^*(v_{i+2r+2}) > 0$, $\tilde{f}^*(v_{i+2j+1}) > 0$ for every j, $0 \le j \le r$, and $\tilde{f}^*(v_{i+2j}) \le 1$ for every j, $1 \le j \le r$. This implies

$$\tilde{f}^*(v_{i+1}) \le 2,\ \tilde{f}^*(v_{i+2r+1}) \le 2,\ \text{and}\ \tilde{f}^*(v_{i+2j+1}) \le 3\ \text{ for every } j,\ 1 \le j \le r-1. \quad (1)$$

   We consider three subcases, according to the number of trunks in σ that are broadcast vertices.

   (a) $\tilde{f}(v_{i+2j}) = 1$ for every j, $1 \le j \le r$.
   In that case, every pendent vertex in σ is an $\tilde{f}$-broadcast vertex, with $\tilde{f}$-value 1. This gives

$$\tilde{f}^*(\sigma) = \lambda(\sigma) + \tau(\sigma) \le 2(r+1) + r = 3r + 2 = 3t_\sigma + 2,$$

   if $v_a \sigma v_b$ is an occurrence of the pattern $1^+2(02)^{+r}1^+$, and

$$\tilde{f}^*(\sigma) = \lambda(\sigma) + \tau(\sigma) \le 1 + 2r + r = 3r + 1 = 3t_\sigma + 1,$$

   otherwise (since we have at least one stem in σ with $\tilde{f}$-value 1).

   (b) $\tilde{f}(v_{i+2j}) = 0$ for every j, $1 \le j \le r$.
   In that case, by (1), we get

$$\tilde{f}^*(\sigma) \le 2 + 3(r-1) + 2 = 3r + 1 = 3t_\sigma + 1.$$

   (c) Not all trunks in σ have the same $\tilde{f}$-value.
   Suppose that $\tilde{f}$ has been chosen in such a way that the number of trunks in σ with $\tilde{f}$-value 0 is maximal. In that case, σ contains two consecutive trunks, say v_{i+2j_0} and v_{i+2j_0+2}, $1 \le j_0 \le r-1$, with $\tilde{f}(v_{i+2j_0}) = 0$ and $\tilde{f}(v_{i+2j_0+2}) = 1$, without loss of generality (by symmetry). This implies $\tilde{f}^*(v_{i+2j_0+1}) = \lambda_{i+2j_0+1} \le 2$. We can then modify $\tilde{f}$ by setting $\tilde{f}(v_{i+2j_0}) = \tilde{f}(v_{i+2j_0+2}) = 0$, $\tilde{f}(\ell^1_{i+2j_0+1}) = 3$ (and $\tilde{f}(\ell^2_{i+2j_0+1}) = 0$ if $\lambda_{i+2j_0+1} = 2$), contradicting our assumption on the maximality of the number of trunks with $\tilde{f}$-value 0. Therefore, this case cannot occur and we are done.

5. The proof uses the same ideas as the proof of the previous case.

   Let $\sigma = v_i v_{i+1} \dots v_{i+2r+2}$ be an occurrence of the pattern $02^-(02^-)^{*r}0$, for some i, $1 \le i \le k - 2r - 3$. We thus have $t_\sigma = r + 2$. Since $\tilde{f}$ satisfies Lemma 13, we have

$$0 < \tilde{f}^*(v_{i+2j+1}) \le 3 \quad \text{for every } j,\ 0 \le j \le r, \tag{2}$$

   and

$$\tilde{f}^*(v_{i+2j}) \le 1 \quad \text{for every } j,\ 0 \le j \le r + 1. \tag{3}$$

   We consider three subcases, according to the number of trunks in σ that are broadcast vertices.

   (a) $\tilde{f}(v_{i+2j}) = 1$ for every j, $0 \le j \le r + 1$.
   In that case, every pendent vertex in σ is an $\tilde{f}$-broadcast vertex, with $\tilde{f}$-value 1. This gives

$$\tilde{f}^*(\sigma) = \lambda(\sigma) + \tau(\sigma) \le 2(r + 1) + r + 2 = 3r + 4 = 3t_\sigma - 2,$$

   if σ is an occurrence of the pattern $02(02)^{*r}0$, and

$$\tilde{f}^*(\sigma) = \lambda(\sigma) + \tau(\sigma) \le 1 + 2r + r + 2 = 3r + 3 = 3t_\sigma - 3,$$

   otherwise (since we have at least one stem in σ with $\tilde{f}$-value 1).

   (b) $\tilde{f}(v_{i+2j}) = 0$ for every j, $0 \le j \le r + 1$.
   In that case, by (2) and (3), we get

$$\tilde{f}^*(\sigma) \le 3(r + 1) = 3r + 3 = 3t_\sigma - 3.$$

   (c) Not all trunks in σ have the same $\tilde{f}$-value.
   Suppose that $\tilde{f}$ has been chosen in such a way that the number of trunks in σ with $\tilde{f}$-value 0 is maximal. In that case, σ contains two consecutive trunks, say v_{i+2j_0} and v_{i+2j_0+2}, $0 \le j_0 \le r$, with $\tilde{f}(v_{i+2j_0}) = 0$ and $\tilde{f}(v_{i+2j_0+2}) = 1$, without loss of generality (by symmetry). This implies $\tilde{f}^*(v_{i+2j_0+1}) = \lambda_{i+2j_0+1} \le 2$. We can then modify $\tilde{f}$ by setting $\tilde{f}(v_{i+2j_0}) = \tilde{f}(v_{i+2j_0+2}) = 0$, $\tilde{f}(\ell^1_{i+2j_0+1}) = 3$ (and $\tilde{f}(\ell^2_{i+2j_0+1}) = 0$ if $\lambda_{i+2j_0+1} = 2$), contradicting our assumption on the maximality of the number of trunks with $\tilde{f}$-value 0. Therefore, this case cannot occur and we are done.

6. Let $v_0 \dots v_{2r+1}$ be an occurrence of the pattern $[2^-(02^-)^{*r}0$ (the case $02^-(02^-)^{*r}]$ is similar, by symmetry). We first prove that for every i, $0 \le i \le r$, $\tilde{f}^*(v_{2i}) + \tilde{f}^*(v_{2i+1}) \le 3$. By Lemma 13, we know that $\tilde{f}(v_{2i+1}) \le 1$. If $\tilde{f}(v_{2i+1}) = 1$, we then have $\tilde{f}(\ell^j_{2i}) \le 1$ for every pendent neighbor ℓ^j_{2i} of v_{2i}, and thus $\tilde{f}^*(v_{2i}) \le \lambda_{2i} \le 2$. On the other hand, if $\tilde{f}(v_{2i+1}) = 0$, we have $\tilde{f}^*(v_{2i}) \le 3$ (which implies $\tilde{f}(\ell^j_{2i}) = 3$ for a pendent neighbor ℓ^j_{2i} of v_{2i}) since otherwise we would have $\tilde{f}^*(v_{2i+2}) = 0$, contradicting Lemma 12. In both cases, we thus get the desired inequality.

   Since σ contains exactly $r + 1 = t_\sigma$ distinct pairs of vertices of the form (v_{2i}, v_{2i+1}), we get

$$\tilde{f}^*(\sigma) = \sum_{i=0}^{i=r} \left(\tilde{f}^*(v_{2i}) + \tilde{f}^*(v_{2i+1}) \right) \le 3(r + 1) = 3t_\sigma.$$

This completes the proof. □

The following lemma states that Lemma 14 covers all possible caterpillars that admit a non-canonical independent broadcast with sufficiently large cost.

Lemma 15 *If $CT = CT(\lambda_0, \dots, \lambda_k)$ is a caterpillar of length $k \geq 1$, with no pair of adjacent trunks, such that there exists an optimal non-canonical independent broadcast f on CT with* $\mathrm{cost}(f) > 2(\mathrm{diam}(CT)-1)$, *then Lemma 14 gives an upper bound on* $\mathrm{cost}(f)$.

Proof. Let $CT = CT(\lambda_0, \dots, \lambda_k)$ be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks, f be an optimal non-canonical independent broadcast on CT with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$, and v_i, $0 \leq i \leq k$, a spine vertex of CT.

If $\lambda_i \geq 3$, then $f^*(v_i) = \lambda_i$ by item 5 of Lemma 11, and thus by item 1 of Lemma 14.

If $\lambda_i = 0$, then $f^*(v_i) \leq 1$ by item 2 of Lemma 13, and thus by item 1 of Lemma 14.

Suppose now that $1 \leq \lambda_i \leq 2$. If $i = 0$ or $i = k$, then $f^*(v_i) \leq 3$ by items 1 to 4 of Lemma 11, and thus by item 1 of Lemma 14. We assume now that $1 \leq i \leq k-1$. If $\lambda_{i-1} > 0$ or $\lambda_{i+1} > 0$, then $f^*(v_i) \leq 2$ by item 3 of Lemma 14.

The remaining case is thus $1 \leq i \leq k-1$, $\lambda_{i-1} = 0$ and $\lambda_{i+1} = 0$. We consider the set of all occurrences of a pattern, in which 0's and 2^-'s alternate, that contain vertices v_{i-1}, v_i and v_{i+1}. Let $\sigma = v_a v_{a+1} \dots v_b$, $0 \leq a \leq i-1 < i+1 \leq b \leq k$ be such an occurrence with maximal length. Note here that we necessarily have $v_a \neq v_i$ and $v_b \neq v_i$. We consider three cases.

1. $\lambda_a = \lambda_b = 0$.
   By the maximality of σ, we necessarily have $\lambda_{a-1} \geq 3$ and $\lambda_{b+1} \geq 3$. Therefore, the value of $f^*(\sigma)$ is bounded by item 5 of Lemma 14.

2. $\lambda_a = 0$ and $\lambda_b > 0$ (the case $\lambda_a > 0$ and $\lambda_b = 0$ is similar, by symmetry).
   By the maximality of σ, we necessarily have $\lambda_{a-1} \geq 3$ and either $b = k$, or $b < k$ and $\lambda_{b+1} \geq 1$. If $b = k$, then the value of $f^*(\sigma)$ is bounded by item 6 of Lemma 14. If $b < k$ and $\lambda_{b+1} \geq 1$, then $f^*(v_a \dots v_{b-1})$ is bounded by item 5 of Lemma 14.

3. $\lambda_a > 0$ and $\lambda_b > 0$.
   By the maximality of σ, we necessarily have (i) either $a = 0$, or $a > 0$ and $\lambda_{a-1} \geq 1$, and (ii) either $b = k$, or $b < k$ and $\lambda_{b+1} \geq 1$.

   If $a > 0$ and $b < k$, then the value of $f^*(\sigma)$ is bounded by item 4 of Lemma 14.

   If $a = 0$ and $b < k$ (the case $a > 0$ and $b = k$ is similar, by symmetry), then the value of $f^*(v_a \dots v_{b-1})$ is bounded by item 6 of Lemma 14.

Finally, if $a = 0$ and $b = k$, the caterpillar CT has pattern $2^-(02^-)^{+r}$. In that case, we have $\mathrm{diam}(CT) = 2r + 2$ and thus $2(\mathrm{diam}(CT) - 1) = 4r + 2$. But by Lemmas 12 and 13 (as discussed in the proof of item 6 of Lemma 14), we have $f^*(v_j)+f^*(v_{j+1}) \leq 3$ for every j, $0 \leq j \leq 2r-2$. Moreover, by item 2 of Lemma 11, we have $f^*(v_{2r}) = 3$. Therefore, $f^*(CT) \leq 3r + 3 \leq 4r + 2 = 2(\mathrm{diam}(CT) - 1)$. This contradicts our assumption on the value of $\mathrm{cost}(f)$, and thus this case cannot occur.

Therefore, in all cases, either $f^*(v_i)$ or $f^*(\sigma)$ for an occurrence σ of a pattern containing v_i is bounded by some item of Lemma 14. This concludes the proof. □

Using Lemmas 14 and 15, we can now prove that no optimal non-canonical independent broadcast f on CT with $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$ and $\mathrm{cost}(f) > \beta^*(CT)$ can exist.

Lemma 16 *Let $CT = CT(\lambda_0, \dots, \lambda_k)$ be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks, and f be an optimal non-canonical independent broadcast on CT with* $\mathrm{cost}(f) > 2(\mathrm{diam}(CT) - 1)$. *We then have* $\mathrm{cost}(f) \leq \beta^*(CT)$.

Proof. Let us denote by f_4 the non-canonical independent broadcast on CT constructed in the proof of Lemma 10, thus with $\mathrm{cost}(f_4) = \beta^*(CT)$.

By considering the four steps involved in the construction of f_4, it clearly appears that f_4 satisfies the five items of Lemma 11, item 2 of Lemma 12 and item 2 of Lemma 13. Therefore, f_4 satisfies item 1 of Lemma 14. Moreover, if v_i is a trunk that does not appear in any pattern considered in Lemma 14, then $f_4(v_i) = 1$. Indeed, the f_4-value of v_i is set to 1 in step 1 of Lemma 10 and is not modified in steps 2 to 4.

We now prove that f_4 satisfies the five last items of Lemma 14 and that, in each case, the upper bound is attained. We will refer to steps 1 to 4 of the proof of Lemma 10 and to the corresponding intermediate independent broadcasts f_1 to f_3. Recall first that in step 1, every trunk and every pendent vertex is assigned the value 1.

1. *Item 2 of Lemma 14.*
   If v_i is a stem with $\lambda_i \geq 3$, the value of its pendent neighbors is not modified in steps 2 to 4. Therefore, we get $f_4^*(v_i) = f_1^*(v_i) = \lambda_i$ for every such v_i.

2. *Item 3 of Lemma 14.*
   Let $v_a v_{a+1}$, $0 \leq a < k$, be an occurrence of the pattern 1^+2^- (the case 2^-1^+ is similar, by symmetry). Note here that if v_{a+1} is the leftmost vertex of an occurrence of the pattern $1^+2(02)^{+r}1^+$, then the value of its pendent neighbors is not modified in step 3.

   If $\lambda_{a+1} = 1$, then, in step 2, the value of ℓ^1_{a+1} is set to 2 and not modified in step 4. If $\lambda_{a+1} = 2$, then the value of the pendent neighbors of v_{a+1} is not modified in steps 2 and 4. Therefore, $f_4^*(v_{a+1}) = 2$ in both cases.

3. *Item 4 of Lemma 14.*
Let $v_a\sigma v_b = v_iv_{i+1}\dots v_{i+2r+2}$ be an occurrence of the pattern $1^+2^-(02^-)^{+r}1^+$, for some i, $0 \le i \le k-2r-2$. In that case, we have $t_\sigma = r$.

If $v_a\sigma v_b$ is an occurrence of the pattern $1^+2(02)^{+r}1^+$, the value of the vertices of σ are not modified in steps 2 to 4. Therefore, we have $f_4^*(\sigma) = f_1^*(\sigma) = 2(r+1)+r = 3r+2 = 3t_\sigma + 2$.

Suppose now that σ contains at least one stem having only one pendent neighbors. In step 3, the value of ℓ^1_{i+1} is set to 2 if $\lambda_{i+1} = 1$, the value of ℓ^1_{i+2r+1} is set to 2 if $\lambda_{i+2r+1} = 1$, the value of ℓ^1_{i+2j+1}, $1 \le j \le r-1$, is set to 3 (and the value of ℓ^2_{i+2j+1} is set to 0 if $\lambda_{i+2j+1} = 2$), and the value of every trunk is set to 0. We thus get

$$f_4^*(\sigma) = f_3^*(\sigma) = 2+2+3(r-1) = 3r+1 = 3t_\sigma + 1.$$

4. *Item 5 of Lemma 14.*
Let $\sigma = v_iv_{i+1}\dots v_{i+2r+2}$ be an occurrence of the pattern $02^-(02^-)^{*r}0$, for some i, $1 \le i \le k-2r-3$. In that case, we have $t_\sigma = r+2$.

If σ is an occurrence of the pattern $02(02)^{*r}0$, the value of the vertices of σ are not modified in steps 2 to 4. Therefore, we have $f_4^*(\sigma) = f_1^*(\sigma) = 2(r+1)+r+2 = 3r+4 = 3t_\sigma - 2$.

Suppose now that σ contains at least one stem having only one pendent neighbor. In step 3, the value of ℓ^1_{i+2j+1}, $0 \le j \le r$, is set to 3 (and the value of ℓ^2_{i+2j+1} is set to 0 if $\lambda_{i+2j+1} = 2$), and the value of every trunk is set to 0. We thus get

$$f_4^*(\sigma) = f_3^*(\sigma) = 3(r+1) = 3r+3 = 3t_\sigma - 3.$$

5. *Item 6 of Lemma 14.*
Let $v_0\dots v_{2r+1}$ be an occurrence of the pattern $[2^-(02^-)^{*r}0$ (the case $02^-(02^-)^{*r}]$ is similar, by symmetry). In that case, we have $t_\sigma = r+1$.

In step 3, the value of ℓ^1_{2j}, $0 \le j \le r$, is set to 3 (and the value of ℓ^2_{2j} is set to 0 if $\lambda_{2j} = 2$), and the value of every trunk is set to 0. We thus get

$$f_4^*(\sigma) = f_3^*(\sigma) = 3(r+1) = 3r+3 = 3t_\sigma.$$

By Lemma 14, we know that there exists an optimal non-canonical independent broadcast $\tilde{f}$ with $\mathrm{cost}(\tilde{f}) = \mathrm{cost}(f)$ which satisfies all items of Lemma 14. We have proved that the non-canonical independent broadcast f_4 constructed in the proof of Lemma 10 also satisfies all items of Lemma 14. Thanks to Lemma 15, we thus have

$$\mathrm{cost}(f) = \mathrm{cost}(\tilde{f}) \le \mathrm{cost}(f_4) = \beta^*(CT),$$

which completes the proof. □

We are now able to state our main result, which determines the broadcast independent number of any caterpillar with no pair of adjacent trunks.

Theorem 17 *Let $CT = CT(\lambda_0, \ldots, \lambda_k)$ be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks. The broadcast independence number of CT is then given by:*

$$\beta_b(CT) = \max\left\{2(\text{diam}(CT) - 1), \beta^*(CT)\right\}.$$

Proof. We know by Observation 1 that $\beta_b(CT) \geq 2(\text{diam}(CT) - 1)$ and we already observed that the canonical independent broadcast f_c on CT satisfies $\text{cost}(f_c) = 2(\text{diam}(CT) - 1)$. According to Lemma 10, it is thus enough to prove that for any optimal non-canonical independent broadcast f on CT with $\text{cost}(f) > 2(\text{diam}(CT) - 1)$, $\text{cost}(f) \leq \beta^*(CT)$, which directly follows from Lemma 16. □

In several cases, the value of $\beta^*(CT)$ has a simple expression. Consider for instance a caterpillar CT, of length $k \geq 1$, having no trunk. We then have $\beta^*(CT) = \lambda(CT) + n_1(CT)$, where n_1 stands for the number of spine vertices having exactly one pendent vertex. Since $\lambda(CT) \geq n_1(CT) + 2(k+1-n_1(CT)) = 2k+2-n_1(CT)$ (spine vertices have either one or at least two pendent neighbors), we get $\beta^*(CT) \geq 2k+2$, with equality if and only if CT contains no stem with at least three pendent neighbors. Since $2(\text{diam}(CT) - 1) = 2k + 2$, we get the following corollary of Theorem 17.

Corollary 18 *Let CT be a caterpillar of length $k \geq 1$ having no trunk. We then have $\beta_b(CT) = 2k + 2 = 2(\text{diam}(CT) - 1)$ if CT has no stem with at least three pendent neighbors, and $\beta_b(CT) = \lambda(CT) + n_1(CT)$ otherwise.*

Moreover, thanks to Observation 4, we can also give the broadcast independent number of caterpillars having adjacent trunks but not stem with at least three pendent neighbors.

Corollary 19 *Let CT be a caterpillar of length $k \geq 1$. If CT has no stem with at least three pendent neighbors, then $\beta_b(CT) = 2k + 2 = 2(\text{diam}(CT) - 1)$.*

Finally, note that if every stem in a caterpillar CT of length $k \geq 1$ with no pair of adjacent trunks has at least three pendent neighbors, then no pattern involved in the definition of $\beta^*(CT)$ can appear in CT. In that case, since $\tau(CT) \leq \left\lfloor \frac{k}{2} \right\rfloor$ and $\lambda(CT) \geq 3\left(\left\lceil \frac{k}{2} \right\rceil + 1\right)$, we get

$$\beta^*(CT) = \lambda(CT) + \tau(CT) > 2k + 2 = 2(\text{diam}(CT) - 1).$$

Therefore, we have:

Corollary 20 *Let CT be a caterpillar of length $k \geq 1$, with no pair of adjacent trunks. If all stems in CT have at least three pendent neighbors, then $\beta_b(CT) = \lambda(CT) + \tau(CT)$.*

References

[1] J.R.S. Blair, P. Heggernes, S. Horton, and F. Manne. Broadcast domination algorithms for interval graphs, series-parallel graphs and trees. *Congr. Num.* 169:55–77 (2004).

[2] I. Bouchemakh and A. Boumali. Broadcast domination number of the cross product of paths. In: ODSA 2010 Conference, Universität Rostock, September 13â15 (2010).

[3] I. Bouchemakh and R. Sahbi. On a conjecture of Erwin. *Stud. Inform. Univ.* 9(2):144–151 (2011).

[4] I. Bouchemakh and M. Zemir. On the Broadcast Independence Number of Grid Graph. *Graphs Combin.* 30:83–100 (2014).

[5] B. Brešar and S. Špacapan. Broadcast domination of products of graphs. *Ars Combin.* 92:303–320 (2009).

[6] E.J. Cockayne, S. Herke and C.M. Mynhardt. Broadcasts and domination in trees. *Discrete Math.* 311(13):1235–1246 (2011).

[7] J. Dabney, B.C. Dean, and S.T. Hedetniemi. A linear-time algorithm for broadcast domination in a tree. *Networks* 53(2):160–169 (2009).

[8] J.E. Dunbar, D.J. Erwin, T.W. Haynes, S.M. Hedetniemi and S.T. Hedetniemi. Broadcasts in graphs. *Discrete Appl. Math.* 154:59–75 (2006).

[9] D.J. Erwin. Cost domination in graphs. PhD Thesis, Western Michigan University (2001).

[10] D.J. Erwin. Dominating broadcasts in graphs. *Bull. Inst. Combin. Appl.* 42:89–105 (2004).

[11] P. Heggernes and D. Lokshtanov. Optimal broadcast domination in polynomial time. *Discrete Math.* 36:3267–3280 (2006).

[12] S. Herke and C.M. Mynhardt. Radial trees. *Discrete Math.* 309:5950–5962 (2009).

[13] S. Lunney and C.M. Mynhardt. More trees with equal broadcast and domination numbers. *Australas. J. Combin.* 61:251–272 (2015).

[14] C.M. Mynhardt and J. Wodlinger. A class of trees with equal broadcast and domination numbers. *Australas. J. Combin.* 56:3–22 (2013).

[15] S.M. Seager. Dominating Broadcasts of Caterpillars. *Ars Combin.* 88:307–319 (2008).

[16] K.W. Soh and K.M. Koh. Broadcast domination in graph products of paths. *Australas. J. Combin.* 59:342–351 (2014).