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Incidence Choosability of Graphs

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Abstract: An incidence of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v, e) and (w, f) of G are adjacent whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f.

An incidence p-colouring of G is a mapping from the set of incidences of G to the set of colours $\{1, \ldots, p\}$ such that every two adjacent incidences receive distinct colours. Incidence colouring has been introduced by Brualdi and Quinn Massey in 1993 and, since then, studied by several authors.

In this paper, we introduce and study the list version of incidence colouring. We determine the exact value of – or upper bounds on – the incidence choice number of several classes of graphs, namely square grids, Halin graphs, cactuses and Hamiltonian cubic graphs.

Keywords: Incidence colouring; Incidence list colouring; List colouring; Square grid;Halin graph; Hamiltonian cubic graph.

1   Introduction

All graphs considered in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively, by $\Delta(G)$ the maximum degree of G, and by $\mathrm{dist}_G(u, v)$ the distance between vertices u and v in G.

A *(proper) colouring* of a graph G is a mapping from $V(G)$ to a finite set of colours such that adjacent vertices are assigned distinct colours. Let L be a *list assignment* of G, that is, a mapping that assigns to every vertex v of G a finite list $L(v)$ of colours. The graph G is *L-list colourable* if there exists a proper colouring λ of G satisfying $\lambda(v) \in L(v)$ for every vertex v of G. The graph G is *k-list colourable*, or *k-choosable*, if, for every list assignment L with $|L(v)| = k$ for every vertex v, G is L-list colourable. The *choice number* $\mathrm{ch}(G)$ of G is then defined as the smallest integer k such that G is k-choosable. List colouring was independently introduced by Vizing [17] and Erdős, Rubin and Taylor [4] (see the surveys by Alon [1], Tuza [16], Kratochvìl, Tuza and Voigt [9], or the monography by Chartrand and Zhang [3, Section 9.2]).

An *incidence* of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v, e) and (w, f) of G are *adjacent* whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f. An *incidence p-colouring* of G is a mapping from the set of incidences of G to the set of colours $\{1, \ldots, p\}$ such that every two adjacent incidences receive distinct colours. The smallest p for which G admits an incidence p-colouring is the *incidence chromatic number* of G, denoted by $\chi_i(G)$. Incidence colourings were first introduced and studied by Brualdi and Quinn Massey [2]. Incidence colourings of various graph families have attracted much interest in recent years, see for instance [5, 6, 8, 10, 15, 18, 19].

The list version of incidence colouring is defined in a way similar to the case of ordinary proper vertex colouring. We thus say that a graph G is *incidence k-choosable*, if, for every list assignment L with $|L(v, e)| = k$ for every incidence (v, e), G is *L-list incidence colourable*. The *incidence choice number* of G, denoted by $\mathrm{ch}_i(G)$, is then defined as the smallest integer k such that G is incidence k-choosable.

Our paper is organised as follows. We first give some preliminary results in Section 2. We then study the incidence choice number of square grids in Section 3, of Halin graphs in Section 4, of cactuses in Section 5, and of Hamiltonian cubic graphs in Section 6. We finally propose some directions for future research in Section 7.

2   Preliminary results

We list in this section some basic results on the incidence choice number of various graph classes. Note first that the inequality $\mathrm{ch}_i(G) \geq \chi_i(G)$ obviously holds for every graph G,

and that whenever G is not connected, $\chi_i(G)$ (resp. $\mathrm{ch}_i(G)$) equals the maximum value of $\chi_i(C)$ (resp. of $\mathrm{ch}_i(C)$), taken over all connected components C of G. Therefore, when studying the incidence chromatic number or the incidence choice number of special graph classes, it is enough to consider the case of connected graphs.

We start by introducing a few notation. With any graph G, we associate the *incidence graph of G*, denoted by I_G, whose vertices are the incidences of G, two incidences being joined by an edge whenever they are adjacent. Clearly, every incidence colouring of G is nothing but a proper vertex colouring of I_G, so that $\chi_i(G) = \chi(I_G)$ and $\mathrm{ch}_i(G) = \mathrm{ch}(I_G)$. Note also that for every subgraph H of G, I_H is a subgraph of I_G. Hence we have:

Observation 1 *For every subgraph H of a graph G, $\chi_i(H) \le \chi_i(G)$ and $\mathrm{ch}_i(H) \le \mathrm{ch}_i(G)$.*

For every vertex v in a graph G, we denote by $A^-(v)$ the set of incidences of the form (v, vu), and by $A^+(v)$ the set of incidences of the form (u, uv) (see Figure 1). We thus have $|A^-(v)| = |A^+(v)| = \deg(v)$ for every vertex v. For every vertex v, the incidences in $A^-(v)$ are called the *internal incidences of v*, and the incidences in $A^+(v)$ are called the *external incidences of v*. The following observation will be useful.

Observation 2 *For every incidence (v, vu), the set of incidences that are adjacent to (v, vu) is $A^-(v) \cup A^+(v) \cup A^-(u)$, whose cardinality is $2\deg_G(v) + \deg_G(u) - 2$.*

Note also that all incidences in $A^-(v)$ must be assigned pairwise distinct colours in every incidence colouring of G and that the colour of any incidence in $A^+(v)$ must be distinct from the colours assigned to the incidences of $A^-(v)$. Moreover, since every incidence has at most $3\Delta(G) - 2$ adjacent incidences by Observation 2 (see Figure 1), we get:

Proposition 3 *For every graph G, $\Delta(G) + 1 \le \chi_i(G) \le \mathrm{ch}_i(G) \le 3\Delta(G) - 1$.*

It was proved in [4, 17] that the choice number also satisfies a Brooks-like theorem, that is, the inequality $\mathrm{ch}(G) \le \Delta(G)$ holds for every graph G which is neither complete nor an odd cycle. Observe that whenever $\Delta(G) \ge 2$, the incidence graph I_G contains a triangle (induced by three incidences of the form (v, vu_1), (v, vu_2) and (u_1, u_1v), $u_1 \ne u_2$) and is non-complete (two incidences of the form (u_1, u_1v) and (u_2, u_2v), $u_1 \ne u_2$, are neither adjacent). On the other hand, if $\Delta(G) = 1$, then G is a union of K_2's, and thus incidence 2-colourable. Hence, Proposition 3 can be slightly improved as follows:

Proposition 4 *For every graph G with $\Delta(G) \ge 2$, $\Delta(G) + 1 \le \chi_i(G) \le \mathrm{ch}_i(G) \le 3\Delta(G) - 2$.*

Incidence (v, vu) : $\star$

Incidences adjacent to (v, vu) : $\diamond$, $\circ$, $\times$

Incidences in $A^+(v)$: $\circ$

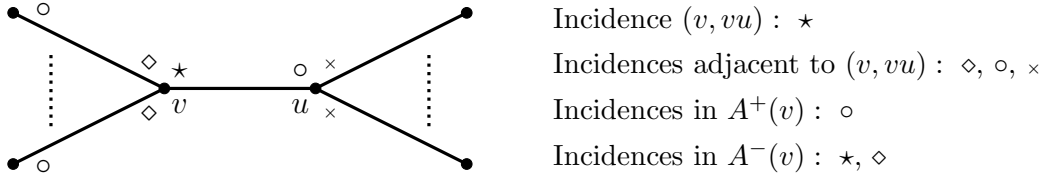Incidences in $A^-(v)$: $\star$, $\diamond$

Figure 1: Adjacent incidences.

Recall that for every integer $p \geq 1$, the pth-power G^p of a graph G is the graph obtained from G by linking every two vertices at distance at most p from each other in G, that is, $V(G^p) = V(G)$ and $uv \in E(G^p)$ if and only if $1 \leq \text{dist}_G(u, v) \leq p$.

Consider now the cycle C_n of order $n \geq 3$. Such a cycle has $2n$ incidences and the associated incidence graph I_{C_n} is the square C_{2n}^2 of the cycle C_{2n}.

In [12], Prowse and Woodall proved that $\text{ch}(C_n^p) = \chi(C_n^p)$ for every $p \geq 1$ and $n \geq 3$, and thus, in particular, for the square of such a cycle. On the other hand, it is not difficult to determine the incidence chromatic number of any cycle C_n [2, 14]. Therefore, we get:

Theorem 5 *For every $n \geq 3$, $3 \leq \text{ch}_i(C_n) = \chi_i(C_n) \leq 4$, with $\text{ch}_i(C_n) = \chi_i(C_n) = 3$ if and only if $n \equiv 0 \pmod 3$.*

A graph G is *d-degenerated* if every subgraph of G contains a vertex of degree at most d. By a simple inductive argument, it is easy to prove that every d-degenerate graph has chromatic number, as well as choice number, at most $d + 1$ [1, Proposition 2.2]. Let v be any vertex of G with degree at most d. Every incidence of the form (v, vu) has then at most $\Delta(G) + 2d - 2$ adjacent incidences in G. Therefore, the incidence graph I_G is $(\Delta(G) + 2d - 2)$-degenerate whenever G is d-degenerate, and we have:

Theorem 6 *For every d-degenerated graph G, $\chi_i(G) \leq \text{ch}_i(G) \leq \Delta(G) + 2d - 1$.*

Since every tree is 1-degenerated, every K_4-minor free graph (and thus every outerplanar graph) is 2-degenerated, and every planar graph is 5-degenerated, Theorem 6 gives the following:

Corollary 7 *For every graph G,*

    1. if G is a tree, then $\text{ch}_i(G) = \Delta(G) + 1$,

    2. if G is a K_4-minor free graph (resp. an outerplanar graph), then $\text{ch}_i(G) \leq \Delta(G)+3$,

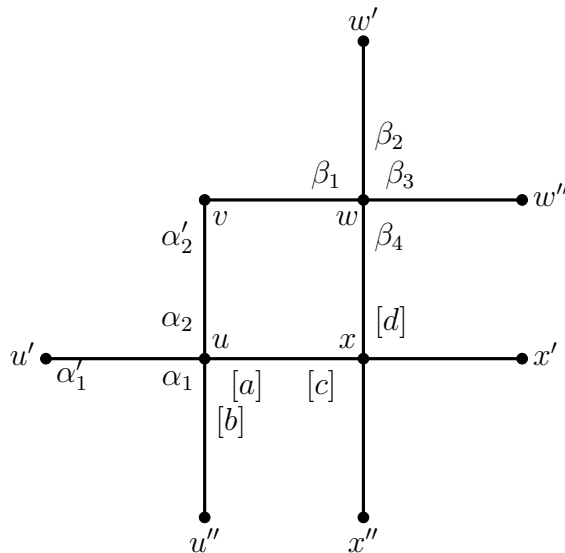   *3. if G is a planar graph, then $\text{ch}_i(G) \leq \Delta(G) + 9$.*

Figure 2: The partially L_0-list incidence coloured graph H_0 of Lemma 8.

3   Square grids

The *square grid* $G_{m,n}$ is the graph defined as the Cartesian product of two paths of respective order m and n, that is, $G_{m,n} = P_m\ P_n$. Since every square grid is 2-degenerated, Theorem 6 gives $\mathrm{ch}_i(G_{m,n}) \leq \Delta(G_{m,n}) + 3 \leq 7$ for every m and n, $m \geq n \geq 2$. In this section, we prove that this bound can be decreased to 5 if $n = 2$ and to 6 if $n \geq 3$.

We first prove the following useful lemma.

Lemma 8 *Let H_0 be the graph depicted in Figure 2, L_0 be any list assignment of H_0 such that $|L_0(i, ij)| \geq 6$ for every incidence (i, ij) of H_0, and σ_0 be the partial L_0-list incidence colouring of H_0 using colours from the set $\{\alpha_1, \alpha'_1, \alpha_2, \alpha'_2\ \beta_1, \beta_2, \beta_3, \beta_4\}$ depicted in Figure 2.*

Then, there exist $a \in L_0(u, ux) \setminus \{\alpha_1, \alpha'_1, \alpha_2, \alpha'_2\}$, $b \in L_0(u, uu'') \setminus \{\alpha_1, \alpha'_1, \alpha_2, \alpha'_2\}$, $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4\}$, and $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4\}$, such that $|\{a, b, c\}| = |\{a, c, d\}| = 3$, so that σ_0 can be extended to colour the four incidences (u, ux), (u, uu''), (x, xu) and (x, xw).

Proof. Note first that $|L_0(u, ux) \setminus \{\alpha_1, \alpha'_1, \alpha_2, \alpha'_2\}| \geq 2$ and $|L_0(u, uu'') \setminus \{\alpha_1, \alpha'_1, \alpha_2, \alpha'_2\}| \geq 2$, so that we can always choose a and b as required.

If $|L_0(x, xu) \cap \{\alpha_1, \alpha_2, \beta_4\}| \leq 2$, then we can choose $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, a\}$ and $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, a, b, d\}$.

Similarly, if $|L_0(x, xw) \cap \{\beta_1, \beta_2, \beta_3, \beta_4\}| \leq 3$, then we can choose $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, a, b\}$ and $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, a, c\}$.

Suppose now that $\{\alpha_1, \alpha_2, \beta_4\} \subseteq L_0(x, xu)$ and $\{\beta_1, \beta_2, \beta_3, \beta_4\} \subseteq L_0(x, xw)$. We consider three cases.

1. If $\beta_4 \in L_0(u, ux) \setminus \{\alpha_1, \alpha_1', \alpha_2, \alpha_2'\}$, then we set $a = \beta_4$. We can then choose $b \in L_0(u, uu'') \setminus \{\alpha_1, \alpha_1', \alpha_2, \alpha_2', \beta_4\}$, $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, b\}$, and $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, c\}$.

2. If $\beta_4 \notin L_0(u, ux) \setminus \{\alpha_1, \alpha_1', \alpha_2, \alpha_2'\}$ and $\beta_4 \in L_0(u, uu'') \setminus \{\alpha_1, \alpha_1', \alpha_2, \alpha_2'\}$, then we set $b = \beta_4$. Again, we can then choose $a \in L_0(u, ux) \setminus \{\alpha_1, \alpha_1', \alpha_2, \alpha_2', \beta_4\}$, $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, a\}$, and $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, a, d\}$.

3. Suppose that none of the previous cases occurs. Let $\{\varepsilon_1, \varepsilon_2\} \subseteq L_0(u, ux) \setminus \{\alpha_1, \alpha_1', \alpha_2, \alpha_2'\}$ and $\{\varepsilon_3, \varepsilon_4\} \subseteq L_0(u, uu'') \setminus \{\alpha_1, \alpha_1', \alpha_2, \alpha_2'\}$. We consider two subcases.

   (a) If $\{\varepsilon_1, \varepsilon_2\} \cap \{\varepsilon_3, \varepsilon_4\} = \emptyset$, we first choose $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4\}$ and $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4\}$ in such a way that $c \neq d$ and $\{\varepsilon_1, \varepsilon_2\} \neq \{c, d\}$ (this can be done since we have at least two choices for d, and then still two choices for c). We then choose $a \in \{\varepsilon_1, \varepsilon_2\} \setminus \{c, d\}$ and $b \in \{\varepsilon_3, \varepsilon_4\} \setminus \{c\}$.

   (b) Otherwise, let $\mu \in \{\varepsilon_1, \varepsilon_2\} \cap \{\varepsilon_3, \varepsilon_4\}$. We consider two subcases.

      i. If $\mu \notin L_0(x, xu)$ or $\mu \notin L_0(x, xw)$, then we set $a = \mu$ and $b \in \{\varepsilon_3, \varepsilon_4\}$ with $b \neq \mu$. Now, if $\mu \notin L_0(x, xu)$, we then choose $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, \mu\}$ and $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, b, d\}$. Otherwise (in which case we have $\mu \notin L_0(x, xw)$), we then choose $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, a, b\}$, and $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, c\}$.

      ii. Suppose finally that $\mu \in L_0(x, xu) \cap L_0(x, xw)$. If $\mu \notin \{\beta_1, \beta_2, \beta_3, \beta_4\}$, then we set $b = d = \mu$ and $a \in \{\varepsilon_1, \varepsilon_2\}$ with $a \neq \mu$. We then choose $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, a, \mu\}$. Otherwise (that is, $\mu \in \{\beta_1, \beta_2, \beta_3, \beta_4\}$), we set $a = \mu$ and $b \in \{\varepsilon_3, \varepsilon_4\}$ with $b \neq \mu$, so that we can choose $c \in L_0(x, xu) \setminus \{\alpha_1, \alpha_2, \beta_4, \mu, b\}$ and $d \in L_0(x, xw) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, c\}$.

In all cases, the colours a, b, c and d clearly satisfy the requirements of the lemma. $\square$

We are now able to prove the main result of this section.

Theorem 9 *For every integers m and n, $m \geq n \geq 2$, we have*

$$\begin{cases} \mathrm{ch}_i(G_{m,n}) \leq 5, & \text{if } n = 2, \\ \mathrm{ch}_i(G_{m,n}) \leq 6, & \text{otherwise.} \end{cases}$$
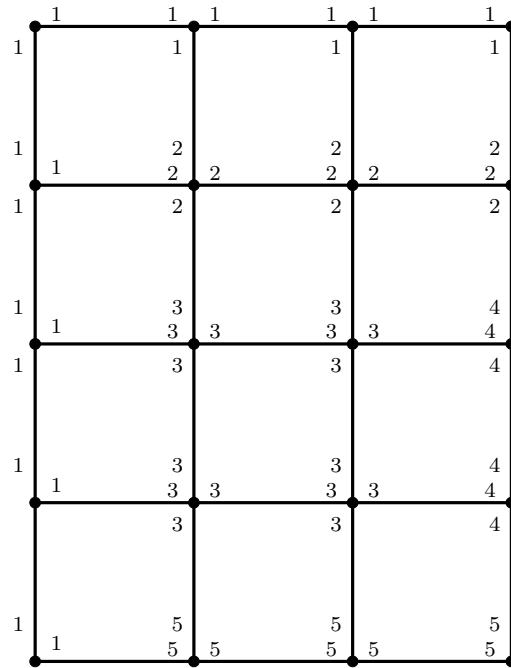
Figure 3: Colouring the incidences of $G_{5,4}$ in five steps (Theorem 9).

Proof. Let $V(G_{m,n}) = \{v_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n\}$, so that $E(G_{m,n}) = \{(v_{i,j}, v_{i',j'}) \mid |i - i'| + |j - j'| = 1\}$.

Suppose first that $n = 2$ and let L be any list assignment of $G_{m,2}$ such that $|L(v, vu)| = 5$ for every incidence (v, vu) of $G_{m,2}$. We construct an L-list incidence colouring of $G_{m,2}$ as follows.

Let us denote by S_i, $1 \le i \le m - 1$, the ith *square* of $G_{m,2}$, that is, the subgraph of $G_{m,2}$ induced by the set of vertices $\{v_{i,1}, v_{i,2}, v_{i+1,1}, v_{i+1,2}\}$. We first colour the incidences of S_1. This can be done since every such incidence has four adjacent incidences.

Then, if $m \ge 3$, we colour the incidences of the remaining squares sequentially, from S_2 to S_{m-1}. For each such square S_i, we colour the incidences $(v_{i,1}, v_{i,1}v_{i+1,1})$, $(v_{i+1,1}, v_{i+1,1}v_{i,1})$, $(v_{i,2}, v_{i,2}v_{i+1,2})$, $(v_{i+1,2}, v_{i+1,2}v_{i,2})$, $(v_{i+1,1}, v_{i+1,1}v_{i+1,2})$ and $(v_{i+1,2}, v_{i+1,2}v_{i+1,1})$, in that order. This can be done since, doing so, every such incidence has at most four already coloured adjacent incidences.

Suppose now that $m \ge n \ge 3$ and let L be any list assignment of $G_{m,n}$ such that $|L(v, vu)| = 6$ for every incidence (v, vu) of $G_{m,n}$. We will construct an L-list incidence colouring of $G_{m,n}$ in five steps. Figure 3 depicts the grid $G_{5,4}$ and gives, for each of its incidences, the number (from 1 to 5) of the step during which it will be coloured.

1. We first colour all internal incidences of vertices $v_{1,j}$, sequentially from $v_{1,1}$ to $v_{1,n}$, and all internal incidences of vertices $v_{i,1}$, sequentially, from $v_{2,1}$ to $v_{m,1}$. This can

be done since, doing so, every such incidence has at most three already coloured adjacent incidences.

2. We then colour all internal incidences of vertices $v_{2,j}$, sequentially from $v_{2,2}$ to $v_{2,n}$. For each such vertex $v_{2,j}$, we colour its internal incidences $(v_{2,j}, v_{2,j}v_{2,j-1})$, $(v_{2,j}, v_{2,j}v_{1,j})$, $(v_{2,j}, v_{2,j}v_{3,j})$ and $(v_{2,j}, v_{2,j}v_{2,j+1})$, in that order (note that $v_{2,n}$ has only the first three internal incidences). This can be done since, doing so, every such incidence has at most five already coloured adjacent incidences.

3. Now, if $m \geq 4$, then, for $i = 2$ to $m - 1$, we colour the uncoloured internal incidences of $v_{i,j}$, sequentially from $v_{i,2}$ to $v_{i,n-1}$ (when $n \geq 4$). Each "row" of internal incidences, corresponding to vertices $v_{i,2}$ to $v_{i,n-1}$, is coloured as follows.

    (a) We colour the internal incidences $(v_{i,2}, v_{i,2}v_{i-1,2})$ and $(v_{i,2}, v_{i,2}v_{i,1})$ of $v_{i,2}$, in that order, which can be done since these two incidences have five already coloured adjacent incidences.

    (b) If $2 \leq j \leq n - 2$, then the set of vertices

    $$\{v_{i,j-1}, v_{i,j}, v_{i,j+1}, v_{i,j+2}, v_{i+1,j}, v_{i+1,j+1}, v_{i-1,j}, v_{i-1,j+1}, v_{i-1,j+2}, v_{i-2,j+1}\}$$

    induces a subgraph of $G_{m,n}$ isomorphic to the graph H_0 of Lemma 8. Therefore, according to Lemma 8, the four incidences $(v_{i,j}, v_{i,j}v_{i,j+1})$, $(v_{i,j}, v_{i,j}v_{i+1,j})$, $(v_{i,j+1}, v_{i,j+1}v_{i,j})$ and $(v_{i,j+1}, v_{i,j+1}v_{i-1,j+1})$ can be coloured with the colours a, b, c and d given by the lemma, respectively.

    (c) We finally colour the two incidences $(v_{i,n-1}, v_{i,n-1}v_{i,n})$ and $(v_{i,n-1}, v_{i,n-1}v_{i+1,n-1})$, in that order, which can be done since, doing so, these incidences have four and five already coloured adjacent incidences, respectively.

4. If $m \geq 4$, we colour all internal incidences of vertices $v_{i,n}$, $3 \leq i \leq m-1$, sequentially from $v_{3,n}$ to $v_{m-1,n}$. For each such vertex $v_{i,n}$, we colour its internal incidences $(v_{i,n}, v_{i,n}v_{i,n-1})$, $(v_{i,n}, v_{i,n}v_{i-1,n})$ and $(v_{i,n}, v_{i,n}v_{i+1,n})$, in that order. This can be done since, doing so, every such incidence has at most five already coloured adjacent incidences.

5. Finally, we colour all (uncoloured) internal incidences of vertices $v_{m,j}$, sequentially from $v_{m,2}$ to $v_{m,n}$. For each such vertex $v_{m,j}$, we colour its internal incidences $(v_{m,j}, v_{m,j}v_{m-1,j})$, $(v_{m,j}, v_{m,j}v_{m,j-1})$ and $(v_{m,j}, v_{m,j}v_{m,j+1})$, in that order (note that $v_{m,n}$ has only the first two internal incidences). This can be done since every such incidence has at most five already coloured adjacent incidences.

This completes the proof.   □

4   Halin graphs

Recall first that the *star* S_n, $n \geq 1$, is the complete bipartite graph $K_{1,n}$. Moreover, the *wheel* W_n, $n \geq 3$, is the graph obtained from the cycle C_n by adding a new vertex adjacent to every vertex of C_n.

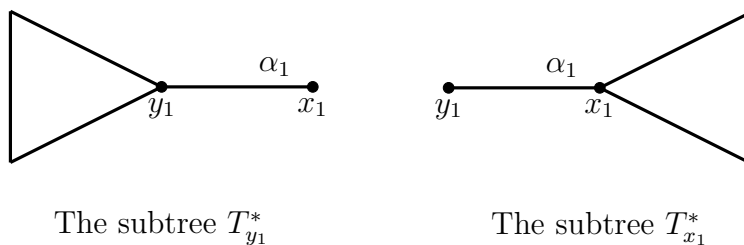The subtree $T^*_{y_1}$ The subtree $T^*_{x_1}$

Figure 4: Configurations for the proof of Proposition 10.

A *Halin graph* is a planar graph obtained from a tree of order at least 4 with no vertex of degree 2, by adding a cycle connecting all its leaves [7]. We call this cycle the *outer cycle of G*. In particular, every wheel is a Halin graph. Wang, Chen and Pang proved that $\chi_i(G) = \Delta(G) + 1$ for every Halin graph G with $\Delta(G) \geq 5$ [18], Shiu and Sun [13] that $\chi_i(G) = 5$ for every cubic Halin graph, and Meng, Guo and Su that $\chi_i(G) \leq \Delta(G) + 2$ for every Halin graph G with $\Delta(G) = 4$ [11].

In this section, we determine the incidence choice number of every Halin graph G with $\Delta(G) \geq 6$ and provide upper bounds for Halin graphs with smaller maximum degree. For every Halin graph G, we denote by C_G the outer cycle of G and by T_G the subgraph of G obtained by deleting all the edges of the outer cycle of G. The subgraph T_G is thus a tree and, in particular, T_G is a star if G is a wheel.

We will prove four lemmas, from which the main result of this section will follow. We first prove a preliminary result, which says that for every tree T and list-assignment L of T with $|L(v, vu)| \geq \Delta(T) + k$ for every incidence (v, vu) of T and some integer $k \geq 1$, one can pre-colour k incidences of T and extend this pre-colouring to an L-list incidence colouring of T.

Proposition 10 *Let T be a tree, $k \geq 1$ be an integer, and L be a list-assignment of T such that $|L(v, vu)| \geq \Delta(T) + k$ for every incidence (v, vu) in T. For every set $\{(x_1, x_1y_1), \ldots, (x_k, x_ky_k)\}$ of k incidences in T and every set $\{\alpha_1, \ldots, \alpha_k\}$ of k colours such that $\alpha_i \in L(x_i, x_iy_i)$ for every i, $1 \leq i \leq k$, and $\alpha_i \neq \alpha_j$ if (x_i, x_iy_i) and (x_j, x_jy_j) are adjacent, $1 \leq i < j \leq k$, there exists an L-list incidence colouring σ of T such that $\sigma(x_i, x_iy_i) = \alpha_i$ for every i, $1 \leq i \leq k$.*

Proof. The proof is by induction on k. Let L be a list-assignment of T with $|L(v, vu)| \geq \Delta(T) + 1$ for every incidence (v, vu) in T, (x_1, x_1y_1) be any incidence in T, and $\alpha_1 \in L(x_1, x_1y_1)$. Let T_{x_1} and T_{y_1} denote the two components (trees) obtained from T by deleting the edge x_1y_1, with $x_1 \in V(T_{x_1})$ and $y_1 \in V(T_{y_1})$. We then denote by $T^*_{x_1}$ and $T^*_{y_1}$ the subtrees of T obtained by adding the edge x_1y_1 to T_{x_1} and T_{y_1}, respectively (see Figure 4), and by L_{x_1} and L_{y_1} the restrictions of L to $T^*_{x_1}$ and $T^*_{y_1}$, respectively. The desired L-list incidence colouring σ of T will be obtained by combining an L_{x_1}-list incidence colouring of $T^*_{x_1}$ with an L_{y_1}-list incidence colouring of $T^*_{y_1}$.

We construct σ_{x_1} as follows. We first set $\sigma_{x_1}(x_1, x_1y_1) = \alpha_1$ and $\sigma_{x_1}(y_1, y_1x_1) = \beta_1$, for some $\beta_1 \in L_{x_1}(y_1, y_1x_1) = L(y_1, y_1x_1)$. Considering y_1 as the root of $T^*_{x_1}$, we can extend σ_{x_1} to an L_{x_1}-list incidence colouring of $T^*_{x_1}$ by colouring the incidences in a top-bottom way, since, doing so, every uncoloured incidence will have at most $\Delta(T^*_{x_1}) \le \Delta(T)$ forbidden colours. The colouring σ_{y_1} is constructed similarly. We first set $\sigma_{y_1}(x_1, x_1y_1) = \alpha_1$ and $\sigma_{y_1}(y_1, y_1x_1) = \beta_1$, and then colour the remaining incidences of $T^*_{y_1}$ in a top-bottom way, considering x_1 as the root of $T^*_{y_1}$. Clearly, combining the colourings σ_{x_1} and σ_{y_1} produces an L-list incidence colouring σ of T with $\sigma(x_1, x_1y_1) = \alpha_1$.

Suppose now that $k > 1$. Let $\{(x_1, x_1y_1), \ldots, (x_k, x_ky_k)\}$ be a set of k incidences in T and $\{\alpha_1, \ldots, \alpha_k\}$ be a set of k colours satisfying the conditions of the proposition. Let L' denote the list assignment of T defined by $L'(v, vu) = L(v, vu) \setminus \{\alpha_k\}$ for every incidence (v, vu) in T. Thanks to the induction hypothesis, there exists an L'-list incidence colouring σ' of T such that $\sigma'(x_i, x_iy_i) = \alpha_i$ for every i, $1 \le i \le k-1$. The required L-list incidence colouring of T is then obtained by setting $\sigma(x_k, x_ky_k) = \alpha_k$ and $\sigma(v, vu) = \sigma'(v, vu)$ for every incidence $(v, vu) \ne (x_k, x_ky_k)$ in T.   □

The next lemma gives a general upper bound on the incidence choice number of Halin graphs. Note that by Proposition 3, the corresponding bound is tight for every Halin graph with maximum degree at least 6.

Lemma 11 *If G is a Halin graph, then* $\mathrm{ch}_i(G) \le \max(\Delta(G) + 1, 7)$.

Proof. Let G be a Halin graph and L be any list assignment of G such that

$$|L(v, vu)| = \max(\Delta(G) + 1, 7) \ge 7$$

for every incidence (v, vu) of G. Let $C_G = v_0v_1 \ldots v_{k-1}v_0$. Each vertex v_i, $0 \le i \le k-1$, has thus three neighbours in G, namely v_{i-1}, v_{i+1} (subscripts are taken modulo k), and some vertex $t_i \in V(T_G) \setminus V(C_G)$ (see Figure 5). Note here that the t_i's are not necessarily distinct. More precisely, we always have $t_i = t_{i-1}$ or $t_i = t_{i+1}$ (or both) for every i, $0 \le i \le k-1$ (subscripts are taken modulo k).

By Corollary 7, we know that T_G is incidence $(\Delta(T_G) + 1)$-choosable, and thus incidence $(\Delta(G) + 1)$-choosable. Let σ be such an L-list incidence colouring of T_G. Since every incidence of C_G has exactly three already coloured adjacent incidences in T_G, and thus at least four available colours in its list, σ can be extended to an L-list incidence colouring of G, thanks to Theorem 5.   □

Using Proposition 10, we can get another upper bound on the incidence choice number of Halin graphs that are not wheels. This new bound thus improves the bound given in Lemma 11 for every Halin graph with maximum degree 3 or 4, except for the two wheels $W_3 = K_4$ and W_4.
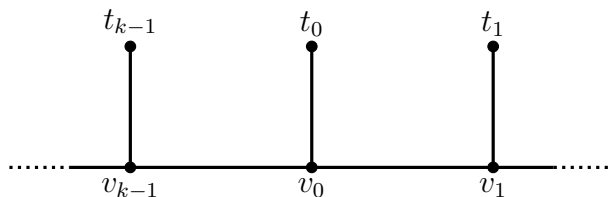
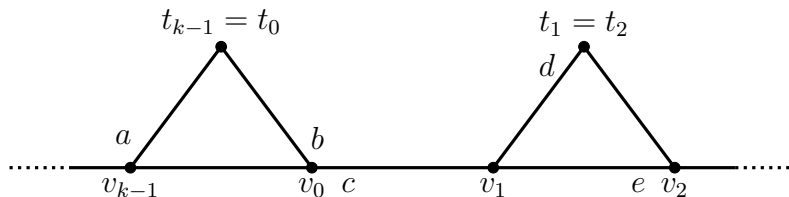Figure 5: Part of the outer cycle C_G of a Halin graph G (the t_i's are not necessarily distinct).

Figure 6: Configuration for the proof of Lemma 12.

Lemma 12 *If G is a Halin graph such that T_G is not a star, then $\mathrm{ch}_i(G) \leq \max(\Delta(G) + 2, 6)$.*

Proof. If $\Delta(G) \geq 5$, the result directly follows from Lemma 11. We can thus assume $\Delta(G) \in \{3, 4\}$ (but we do not need this assumption in the proof).

Let G be a Halin graph and L be any list assignment of G such that

$$|L(v, vu)| = \max(\Delta(G) + 2, 6) \geq 6$$

for every incidence (v, vu) of G, and let $p = \max(\Delta(G) + 2, 6)$. As in the proof of Lemma 11, we let $C_G = v_0 v_1 \ldots v_{k-1} v_0$ and t_i denotes the unique neighbour of v_i in $V(T_G) \setminus V(C_G)$, $0 \leq i \leq k - 1$. Note that starting from an L-list incidence colouring σ of T_G and then colouring the incidences of C_G in cyclic order, starting from any incidence, all incidences of C_G but the last two ones can be coloured, as each of these incidences has at most five forbidden colours. We will prove that one can always fix the colour of some incidences, so that one can produce an L-list incidence colouring of G.

Since T_G is not a star, there exists an index i, $0 \leq i \leq k-1$, such that the vertices t_{i-1} and t_i are distinct. We can thus assume, without loss of generality, that $t_0 \neq t_1$. Moreover, since T_G has no vertex of degree two, we have $t_{k-1} = t_0$ and $t_2 = t_1$ (see Figure 6).

The following claim will be essential in the construction of an L-list incidence colouring of G.

Claim 1 *There exist* $a \in L(v_{k-1}, v_{k-1}t_0)$, $b \in L(v_0, v_0t_0)$, $c \in L(v_0, v_0v_1)$, $d \in L(t_1, t_1v_1)$ *and* $e \in L(v_2, v_2v_1)$, *with* $b \neq c$, *such that*

$$|L(v_{k-1}, v_{k-1}v_0) \cap \{a, b, c\}| \leq 2, \ |L(v_0, v_0v_{k-1}) \cap \{a, b, c\}| \leq 2, \ and \ |L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1.$$

Proof. We first deal with the incidence (v_1, v_1v_0) and set the values of c, d and e. Let $C = L(v_0, v_0v_1)$, $D = L(t_1, t_1v_1)$ and $E = L(v_2, v_2v_1)$. If $C \cap D \cap E \neq \emptyset$, then we set $c = d = e = \gamma$ for some $\gamma \in C \cap D \cap E$, so that $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$.

Otherwise, we have two cases to consider.

1. If C, D and E are pairwise disjoint, then at least two of them are distinct from $L(v_1, v_1v_0)$, so that we can choose c, d and e in such a way that $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$.

2. Suppose now that $C \cap D \neq \emptyset$ (the cases $C \cap E \neq \emptyset$ and $D \cap E \neq \emptyset$ are similar). We first set $c = d = \gamma$ for some $\gamma \in C \cap D$. If $\gamma \in L(v_1, v_1v_0)$, then there exists $\varepsilon \in E \setminus L(v_1, v_1v_0)$ (since $(C \cap D) \cap E = \emptyset$) and we set $e = \varepsilon$, so that $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$. If $\gamma \notin L(v_1, v_1v_0)$, then we set $e = \varepsilon$ for any $\varepsilon \in E$ and we also get $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$.

We now consider the incidence (v_0, v_0v_{k-1}). Let $A = L(v_{k-1}, v_{k-1}t_0)$ and $B = L(v_0, v_0t_0)$. If $c \notin L(v_0, v_0v_{k-1})$, then $|L(v_0, v_0v_{k-1}) \cap \{a, b, c\}| \leq 2$ for any values of a and b.

Suppose now that $c \in L(v_0, v_0v_{k-1})$. If $|A \cap B| \geq 2$, then we set $a = b = \lambda$ for some $\lambda \in (A \cap B) \setminus \{c\}$, so that $|L(v_0, v_0v_{k-1}) \cap \{a, b, c\}| \leq 2$. Otherwise, we necessarily have $A \neq L(v_0, v_0v_{k-1})$ or $B \neq L(v_0, v_0v_{k-1})$. In the former case, we set $a = \alpha$ for some $\alpha \in A \setminus L(v_0, v_0v_{k-1})$, so that $|L(v_0, v_0v_{k-1}) \cap \{a, b, c\}| \leq 2$ for any value of b. In the latter case, we set $b = \beta$ for some $\beta \in B \setminus L(v_0, v_0v_{k-1})$, so that $|L(v_0, v_0v_{k-1}) \cap \{a, b, c\}| \leq 2$ for any value of a.

We finally consider the incidence $(v_{k-1}, v_{k-1}v_0)$. If $c \notin L(v_{k-1}, v_{k-1}v_0)$, then $|L(v_{k-1}, v_{k-1}v_0) \cap \{a, b, c\}| \leq 2$ for any values of a and b and we are done.

Suppose now that $c \in L(v_{k-1}, v_{k-1}v_0)$. If none of the values of a and b have been set in the previous step, then we proceed as for the incidence (v_0, v_0v_{k-1}) and the result follows. Otherwise, we have two cases to consider.

1. If the values of both a and b have been set in the previous step, then we have $a = b = \lambda$, so that $|L(v_{k-1}, v_{k-1}v_0) \cap \{a, b, c\}| \leq 2$.

2. Suppose now that the value of a has been set in the previous step, that is, $a = \alpha$ for some $\alpha \in A \setminus L(v_0, v_0v_{k-1})$ (the proof is similar if the value of b has been set).

If $\alpha \in B$, then we can set $b = \alpha$ and we are done. If $\alpha \notin B$ and $\alpha \notin L(v_{k-1}, v_{k-1}v_0)$, then we get $|L(v_{k-1}, v_{k-1}v_0) \cap \{a, b, c\}| \leq 2$ for any value of b. Otherwise, we have $\alpha \notin B$ and $\alpha \in L(v_{k-1}, v_{k-1}v_0)$, which implies $B \neq L(v_{k-1}, v_{k-1}v_0)$. Therefore, we can set $b = \beta$ for some $\beta \in B \backslash L(v_{k-1}, v_{k-1}v_0)$, so that $|L(v_{k-1}, v_{k-1}v_0) \cap \{a, b, c\}| \leq 2$.

This concludes the proof of Claim 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

We now construct an L-list incidence colouring σ of G in three steps.

1. We first set $\sigma(v_{k-1}, v_{k-1}t_0) = a$, $\sigma(v_0, v_0t_0) = b$, $\sigma(v_0, v_0v_1) = c$, $\sigma(t_1, v_1t_1) = d$, and $\sigma(v_2, v_1v_2) = e$, where a, b, c, d and e are the values determined in the proof of Claim 1.

2. Let $P = t_0u_1 \dots u_\ell t_1$, or $P = t_0t_1$ if $t_0t_1 \in E(G)$, denote the unique path from t_0 to t_1 in T_G (see Figure 7). We colour all the incidences of T_G as follows.

   - We first colour all internal incidences of t_0, starting with the incidence (t_0, t_0v_0), and then the incidence (t_0, t_0t_1) if $t_0t_1 \in E(G)$. This can be done since every such incidence has at most $\Delta(G) + 1$ already coloured adjacent incidences.

   - If $t_0t_1 \notin E(G)$), then we colour the internal incidences of the vertices of P sequentially, from u_1 to u_ℓ. Again, every such incidence has at most $\Delta(G) + 1$ already coloured adjacent incidences.

   - We colour the incidence (t_1, t_1u_ℓ) (or (t_1, t_1t_0) if $t_0t_1 \in E(G)$), which has at most $\Delta(G) + 1$ already coloured adjacent incidences, then the incidence (v_1, v_1t_1), which has four already coloured adjacent incidences, and then the incidence (t_1, t_1v_2), which has five already coloured adjacent incidences (recall that $p \geq 6$).

   - We colour the remaining uncoloured internal incidences of t_1, if any. This can be done since every such incidence has at most $\Delta(G) + 1$ already coloured adjacent incidences.

   - Now, we colour the uncoloured external incidences of the vertices of P, sequentially, from t_0 to t_1. Again, this can be done since every such incidence has at most $\Delta(G) + 1$ already coloured adjacent incidences.

   - For every edge $xy \in E(T_G)$, we denote by T_{xy} the unique maximal subtree of T_G containing the edge xy and such that $\deg_{T_{xy}}(x) = 1$. Clearly, each remaining uncoloured incidence of T_G belongs to some subtree T_{xy}, with $x \in V(P)$ and $y \notin V(P) \cup \{v_{k-1}, v_0, v_1, v_2\}$. Moreover, the only already coloured incidences of any such subtree T_{xy} are (x, xy) and (y, yx). By Proposition 10, we can therefore extend σ to all incidences of T_G.

3. We finally colour all the uncoloured incidences of C_G (the only incidences of C_G already coloured are (v_0, v_0v_1) and (v_2, v_2v_1)) as follows.

   - We first colour the incidence (v_1, v_1v_2), which has five already coloured adjacent incidences.
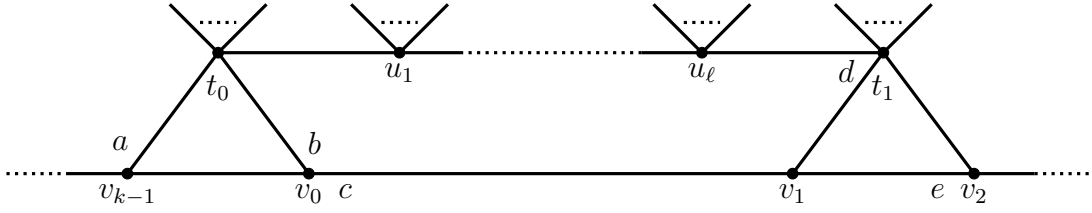
Figure 7: Colouring the subtree of T_G in the proof of Lemma 12.

- We then cyclically colour the incidences of C_G from (v_2, v_2v_3) to $(v_{k-1}, v_{k-1}v_{k-2})$. This can be done since, doing so, every such incidence has four or five already coloured adjacent incidences.

- By Claim 1, the incidence $(v_{k-1}, v_{k-1}v_0)$ has at most five forbidden colours and can thus be coloured. Similarly, thanks to Claim 1, we can also colour the incidences (v_0, v_0v_{k-1}) and (v_1, v_1v_0) (in that order).

This completes the proof.   □

The next lemma shows that the incidence choice number of K_4 is at most 6.

Lemma 13 $\mathrm{ch}_i(K_4) \leq 6$.

Proof. Let $V(K_4) = \{v_0, v_1, v_2, v_3)$ and L be any list assignment of K_4 such that $|L(v_i, v_iv_j)| = 6$ for every incidence (v_i, v_iv_j) of K_4.

The following claim will be useful for constructing an L-list incidence colouring of K_4.

Claim 2 *There exist* $a \in L(v_1, v_1v_0)$, $b \in L(v_2, v_2v_0)$, *and* $c \in L(v_3, v_3v_0)$ *such that*

$$|L(v_0, v_0v_1) \cap \{a, b, c\}| \leq 1.$$

Proof. Let $A = L(v_1, v_1v_0)$, $B = L(v_2, v_2v_0)$ and $C = L(v_3, v_3v_0)$. If $A \cap B \cap C \neq \emptyset$, then we set $a = b = c = \gamma$ for some $\gamma \in A \cap B \cap C$, so that $|L(v_0, v_0v_1) \cap \{a, b, c\}| \leq 1$.

Otherwise, we consider two cases.

1. If A, B and C are pairwise disjoint, then at least two of them are distinct from $L(v_0, v_0v_1)$, so that we can choose a, b and c in such a way that $|L(v_0, v_0v_1) \cap \{a, b, c\}| \leq 1$.

2. Suppose now that $A \cap B \neq \emptyset$ (the cases $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$ are similar). We first set $a = b = \gamma$ for some $\gamma \in A \cap B$. If $\gamma \in L(v_0, v_0v_1)$, then there exists $\varepsilon \in C \setminus L(v_0, v_0v_1)$ (since $(A \cap B) \cap C = \emptyset$) and we set $c = \varepsilon$, so that $|L(v_0, v_0v_1) \cap \{a, b, c\}| \leq 1$. If $\gamma \notin L(v_0, v_0v_1)$, then we set $e = \varepsilon$ for any $\varepsilon \in C$ and we also get $|L(v_0, v_0v_1) \cap \{a, b, c\}| \leq 1$.

This concludes the proof of Claim 2. $\square$

We now construct an L-list incidence colouring σ of K_4, by setting first $\sigma(v_1, v_1v_0) = a$, $\sigma(v_2, v_2v_0) = b$ and $\sigma(v_3, v_3v_0) = c$, where a, b and c are the values determined in the proof of Claim 2.

We then consider two cases.

1. Suppose first that $|\{a, b, c\}| \leq 2$ and assume $a = b$ (the cases $a = c$ and $b = c$ are similar). We then colour the remaining uncoloured incidences as follows (see Figure 8(a)). We first colour the incidences (v_3, v_3v_1), (v_3, v_3v_2), (v_2, v_2v_3), (v_1, v_1v_3) and (v_2, v_2v_1), in that order. This can be done since, doing so, every such incidence has at most five already coloured adjacent incidences. We then colour the incidences (v_1, v_1v_2), (v_0, v_0v_3) and (v_0, v_0v_2), in that order. This can be done since, doing so, every such incidence has at most five forbidden colours (recall that $a = b$). We finally colour the incidence (v_0, v_0v_1), which has at least one available colour in its own list since, by Claim 2, $|L(v_0, v_0v_1) \cap \{a, b, c\}| \leq 1$).

2. Suppose now that $|\{a, b, c\}| = 3$. By symmetry and thanks to Claim 2, we may assume $L(v_0, v_0v_1) \cap \{a, b\} = \emptyset$, without loss of generality. We consider two subcases.

   (a) $|L(v_0, v_0v_2) \cap \{a, b\}| \leq 1$ (or, similarly, $|L(v_0, v_0v_3) \cap \{a, b\}| \leq 1$).
   We first uncolour the incidence (v_3, v_3v_0) (note that for any choice of $\sigma(v_3, v_3v_0)$, the statement of Claim 2 will be satisfied). We then colour the remaining uncoloured incidences as follows (see Figure 8(b)). We first colour the incidences (v_1, v_1v_3), (v_1, v_1v_2), (v_2, v_2v_1), (v_2, v_2v_3), (v_3, v_3v_2), (v_3, v_3v_1), (v_3, v_3v_0) and (v_0, v_0v_3), in that order. This can be done since, doing so, every such incidence has at most five already coloured adjacent incidences. We then colour the incidence (v_0, v_0v_2), which has at most five forbidden colours since $|L(v_0, v_0v_2) \cap \{a, b\}| \leq 1$, and the incidence (v_0, v_0v_1), which has also at most five forbidden colours since $|L(v_0, v_0v_1) \cap \{a, b, \sigma(v_3, v_3v_0)\}| \leq 1$.

   (b) $\{a, b\} \subseteq (L(v_0, v_0v_2) \cap L(v_0, v_0v_3))$.
   We first uncolour the incidences (v_1, v_1v_0) and (v_2, v_2v_0), and set $\sigma(v_0, v_0v_2) = a$ and $\sigma(v_0, v_0v_3) = b$ (this is possible since $c \notin \{a, b\}$).
   We claim that there exists a colour $d \in L(v_1, v_1v_2)$ such that $|L(v_1, v_1v_0) \cap \{b, d\}| \leq 1$. This is obviously the case if $b \notin L(v_1, v_1v_0)$. Assume thus that $b \in L(v_1, v_1v_0)$. If $b \in L(v_1, v_1v_2)$, then we can set $d = b$. Otherwise, it suffices to choose any d in $L(v_1, v_1v_2) \setminus L(v_1, v_1v_0)$. We then set $\sigma(v_1, v_1v_2) = d$.

(a) Case 1 ($a = b$)   (b) Case 2(a)
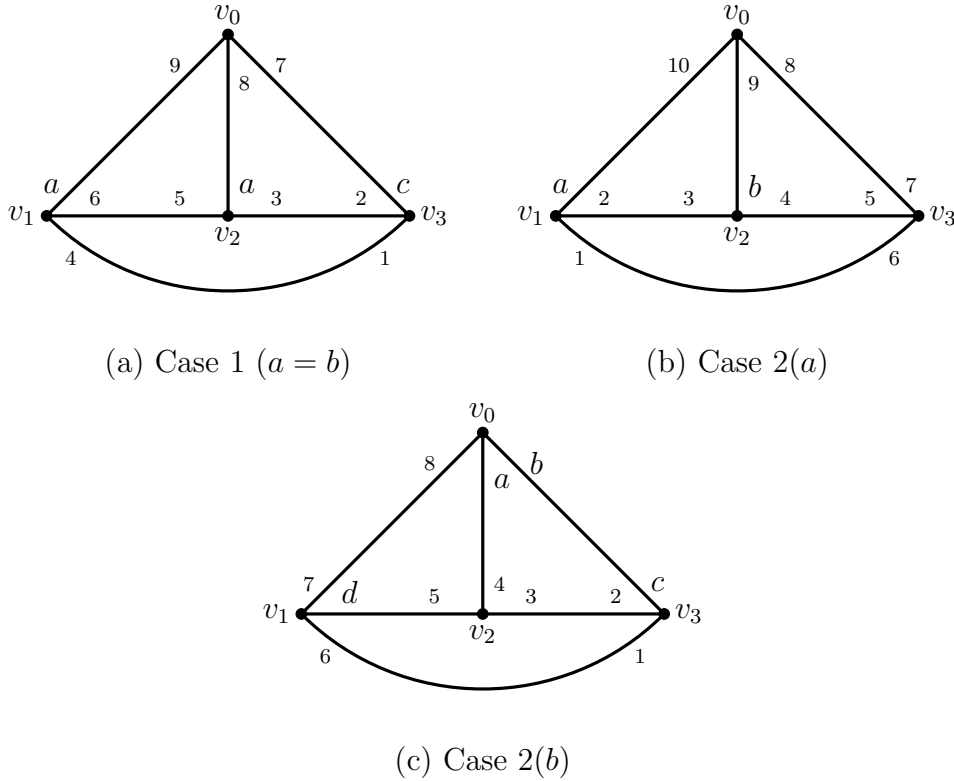

(c) Case 2(b)

Figure 8: Ordering the incidences of K_4 for the proof of Lemma 13.

We then colour the remaining uncoloured incidences as follows (see Figure 8(c)). We first colour the incidences (v_3, v_3v_1), (v_3, v_3v_2), (v_2, v_2v_3), (v_2, v_2v_0), (v_2, v_2v_1) and (v_1, v_1v_3), in that order. This can be done since, doing so, every such incidence has at most five already coloured adjacent incidences. We then colour the incidences (v_1, v_1v_0), which has at most five forbidden colours since $|L(v_1, v_1v_0) \cap \{b, d\}| \leq 1$), and (v_0, v_0v_1), which has also at most five forbidden colours since $L(v_0, v_0v_1) \cap \{a, b\} = \emptyset$.

This completes the proof.   □

By Proposition 3 and Lemmas 11, 12 and 13, we get:

Theorem 14 *If G is a Halin graph, then*

$$
\begin{cases}
\mathrm{ch}_i(G) \leq 6, & \text{if } \Delta(G) \in \{3, 4\} \text{ and } G \neq W_4, \\
\mathrm{ch}_i(G) \leq 7, & \text{if } \Delta(G) = 5 \text{ or } G = W_4, \\
\mathrm{ch}_i(G) = \Delta(G) + 1, & \text{otherwise.}
\end{cases}
$$

5   Cactuses

A *cactus* is a (planar) graph such that every vertex belongs to at most one cycle. The *corona* $G \odot K_1$ of a graph G is the graph obtained from G by adding one pendent neighbour to each vertex of G. A *generalized corona* of a graph G is a graph $G \odot pK_1$, for some integer $p \geq 1$, obtained from G by adding p pendent neighbours to each vertex of G. In particular, every generalized corona of a cycle is thus a cactus.

We give in this section an upper bound on the incidence choice number of cactuses. In order to do that, we will first consider the case of generalized coronae of cycles.

For every integer $n \geq 3$, we let $V(C_n) = \{v_0, \dots, v_{n-1}\}$. For every generalized corona $C_n \odot pK_1$ of the cycle C_n and every vertex v_i of C_n, $0 \leq i \leq n-1$, we denote by $v_i^1, \dots, v_i^p$ the p pendent neighbours of v_i.

Let $G = C_n \odot pK_1$, with $n \geq 3$ and $p \geq 1$, be a generalized corona of C_n, and L be any list assignment of G such that $|L(v, vu)| = \Delta(G) + 2$ for every incidence (v, vu) of G. By colouring first the incidences of C_n, then the uncoloured internal incidences of $v_0, \dots, v_{n-1}$, and finally the external incidences of $v_0, \dots, v_{n-1}$, we can produce an L-list incidence colouring of G since, doing so, every incidence has at most $\Delta(G) + 1$ already coloured adjacent incidences. Therefore, $\mathrm{ch}_i(C_n \odot pK_1) \leq \Delta(C_n \odot pK_1) + 2$ for every generalized corona $C_n \odot pK_1$.

The next lemma shows that we can decrease by 1 this bound whenever $p \geq 4$. Note that by Proposition 3, in that case, the corresponding bound is tight. Since it will be useful for studying the incidence choice number of cactuses, the next lemma also considers the case when the two incidences of one pendent edge are pre-coloured, and proves that an additional colour is needed in that case only when $n = 3$ and $p \geq 3$.

Lemma 15 *For every integers $n \geq 3$ and $p \geq 1$,*

$$\mathrm{ch}_i(C_n \odot pK_1) \leq \begin{cases} \Delta(C_n \odot pK_1) + 2 = p + 4, & \text{if } p \leq 2, \\ \max(\Delta(C_n \odot pK_1) + 1, 7) = \max(p + 3, 7), & \text{otherwise.} \end{cases}$$

Moreover, for every list assignment L of $C_n \odot pK_1$ with $|L(v, vu)| = k$ for every incidence (v, vu) of $C_n \odot pK_1$, $a \in L(v_0, v_0 v_0^1)$ and $b \in L(v_0^1, v_0^1 v_0)$, $a \neq b$, there exists an L-incidence colouring σ of $C_n \odot pK_1$ with $\sigma(v_0, v_0 v_0^1) = a$ and $\sigma(v_0^1, v_0^1 v_0) = b$ in each of the following cases:

1. *$p \leq 2$ and $k \geq p + 4$,*

2. *$n > 3$, $p \geq 3$ and $k \geq \max(p + 3, 7)$,*

3. *$n = 3$, $p \geq 3$ and $k \geq \max(p + 3, 8)$.*

Proof. Since the proof when two incidences are pre-coloured is similar to the proof of the general bound, we give these two proofs simultaneously, referring to the former case as the pre-coloured case. In the following, subscripts are always taken modulo n.

We first consider the case $p \leq 2$. Let L be any list assignment of $C_n \odot pK_1$ such that $|L(v, vu)| = p + 4$ if $p \leq 2$ for every incidence (v, vu) of $C_n \odot pK_1$, and let $a \in L(v_0, v_0v_0^1)$ and $b \in L(v_0^1, v_0^1v_0)$, $a \neq b$. We will construct an L-list incidence colouring σ of $C_n \odot pK_1$ in three steps. We first set $\sigma(v_0, v_0v_0^1) = a$ and $\sigma(v_0^1, v_0^1v_0) = b$, even if we are not in the pre-coloured case.

1. *Incidences of C_n.*
   If $p = 1$, there is only one edge incident to v_0 not belonging to C_n, and both its incidences are already coloured. If $p = 2$, we claim that there exists $c \in L(v_{n-1}, v_{n-1}v_0)$ such that $|L(v_0, v_0v_0^2) \cap \{a, b, c\}| \leq 2$ and we set $\sigma(v_{n-1}, v_{n-1}v_0) = c$. Indeed, if $\{a, b\} \not\subseteq L(v_0, v_0v_0^2)$, then $|L(v_0, v_0v_0^2) \cap \{a, b, c\}| \leq 2$ for any value of $c \in L(v_{n-1}, v_{n-1}v_0)$. Suppose now that $\{a, b\} \subseteq L(v_0, v_0v_0^2)$. If $b \in L(v_{n-1}, v_{n-1}v_0)$, then we set $c = b$. Otherwise, we set $c = \gamma$ for some $\gamma \in L(v_{n-1}, v_{n-1}v_0) \setminus L(v_0, v_0v_0^2)$.

   We then colour the remaining uncoloured incidences of C_n cyclically, from (v_0, v_0v_{n-1}) to $(v_{n-1}, v_{n-1}v_{n-2})$, which can be done since, doing so, every such incidence has at most $4 < p + 4$ already coloured adjacent incidences.

2. *Uncoloured internal incidences of v_i, $0 \leq i \leq n - 1$.*
   If $p = 2$, we colour the incidence $(v_0, v_0v_0^2)$, which can be done since it has at most 5 forbidden colours (recall that $|L(v_0, v_0v_0^2) \cap \{\sigma(v_0, v_0v_0^1), \sigma(v_0^1, v_0^1v_0), \sigma(v_{n-1}, v_{n-1}v_0)\}| \leq 2$ thanks to the previous step).

   Now, for each vertex v_i, $1 \leq i \leq n - 1$, we colour the incidence $(v_i, v_iv_i^1)$, or the incidences $(v_i, v_iv_i^1)$ and $(v_i, v_iv_i^2)$, in that order, if $p = 2$. This can be done since, doing so, every such incidence $(v_i, v_iv_i^j)$, $1 \leq j \leq p$, has $j + 3 < p + 4$ already coloured adjacent incidences.

3. *External incidences of v_i, $0 \leq i \leq n - 1$.*
   We finally colour all uncoloured incidences of the form $(v_i^j, v_i^jv_i)$, $0 \leq i \leq n - 1$, $1 \leq j \leq p$, which can be done since every such incidence has at most $p + 2$ already coloured adjacent incidences.

The above-constructed mapping σ is clearly an L-list incidence colouring σ of $C_n \odot pK_1$ with $\sigma(v_0, v_0v_0^1) = a$ and $\sigma(v_0^1, v_0^1v_0) = b$, as required.

We now consider the case $p \geq 3$. Let L be any list assignment of $C_n \odot pK_1$ such that, for every incidence (v, vu) of $C_n \odot pK_1$, $|L(v, vu)| = \max(p + 3, 7)$ if we are not in the pre-coloured case or $n > 3$, and $|L(v, vu)| = \max(p + 3, 8)$ otherwise. Moreover, if we are in the pre-coloured case, then let $a \in L(v_0, v_0v_0^1)$ and $b \in L(v_0^1, v_0^1v_0)$, $a \neq b$.

We will construct an L-list incidence colouring σ of $C_n \odot pK_1$ in two steps. If we are in the pre-coloured case, we first set $\sigma(v_0, v_0v_0^1) = a$ and $\sigma(v_0^1, v_0^1v_0) = b$.
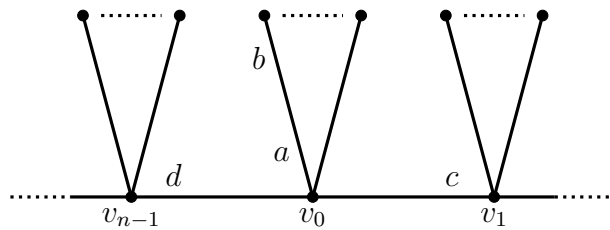
Figure 9: Configuration for the proof of Lemma 15, pre-coloured case.

1. *Incidences of C_n.*

   We first construct a partial L-list incidence colouring σ_0 of $C_n \odot pK_1$, fixing the colour of all incidences of C_n, and satisfying the following property:

   (P) For every i, $0 \leq i \leq n-1$ (or $1 \leq i \leq n-1$ if we are in the pre-coloured case),

   $$|L(v_i, v_i v_i^p) \cap \{\sigma_0(v_{i-1}, v_{i-1} v_i), \sigma_0(v_{i+1}, v_{i+1} v_i)\}| \leq 1.$$

   Moreover, if we are in the pre-coloured case, then

   $$|L(v_0, v_0 v_0^p) \cap \{a, b, \sigma_0(v_{n-1}, v_{n-1} v_0), \sigma_0(v_1, v_1 v_0)\}| \leq 2.$$

   We proceed in two steps.

   (a) If we are in the pre-coloured case, then we first claim that there exist $c \in L(v_1, v_1 v_0)$ and $d \in L(v_{n-1}, v_{n-1} v_0)$, $c \neq a$, $d \neq a$, such that $|L(v_0, v_0 v_0^p) \cap \{a, b, c, d\}| \leq 2$, and set $\sigma_0(v_1, v_1 v_0) = c$ and $\sigma_0(v_{n-1}, v_{n-1} v_0) = d$ (see Figure 9). To see that, we consider two cases.

      i. $|\{a, b\} \cap L(v_0, v_0 v_0^p)[\leq 1$.

         In that case, it suffices to choose c and d in such a way that $|\{c, d\} \cap L(v_0, v_0 v_0^p)| \leq 1$. This can be done since either $(L(v_{n-1}, v_{n-1} v_0) \cap L(v_1, v_1 v_0)) \setminus \{a\} \neq \emptyset$, in which case we choose $c = d = \gamma$ for some $\gamma \in (L(v_{n-1}, v_{n-1} v_0) \cap L(v_1, v_1 v_0)) \setminus \{a\}$, or $(L(v_{n-1}, v_{n-1} v_0) \cap L(v_1, v_1 v_0)) \setminus \{a\} = \emptyset$, which implies

         $$|L(v_{n-1}, v_{n-1} v_0) \cup L(v_1, v_1 v_0)| \geq \max(2(p+2), 12),$$

         and we can choose c and d in such a way that $|\{c, d\} \cap L(v_0, v_0 v_0^p)[\leq 1$.

      ii. $\{a, b\} \subseteq L(v_0, v_0 v_0^p)$.

         If $b \in L(v_1, v_1 v_0)$, then we set $c = b$. Otherwise, we set $c = \gamma$ for some $\gamma \in L(v_1, v_1 v_0) \setminus L(v_0, v_0 v_0^p)$. Similarly, if $b \in L(v_{n-1}, v_{n-1} v_0)$ then we set $d = b$. Otherwise, we set $d = \delta$ for some $\delta \in L(v_{n-1}, v_{n-1} v_0) \setminus L(v_0, v_0 v_0^p)$.

      In all cases, we get $|L(v_0, v_0 v_0^p) \cap \{a, b, c, d\}| \leq 2$.

      In both cases (pre-coloured or not), we are going to colour some incidences of C_n, in such a way that for every i, $0 \leq i \leq n-1$ (or $1 \leq i \leq n-1$ if we are in the pre-coloured case), we have the following property:

      (P') Either $\sigma_0(v_{i-1}, v_{i-1} v_i) = \sigma_0(v_{i+1}, v_{i+1} v_i)$, or one of $\sigma_0(v_{i-1}, v_{i-1} v_i)$, $\sigma_0(v_{i+1}, v_{i+1} v_i)$ only is set and, in that case, the assigned colour does not belong to $L(v_i, v_i v_i^p)$.
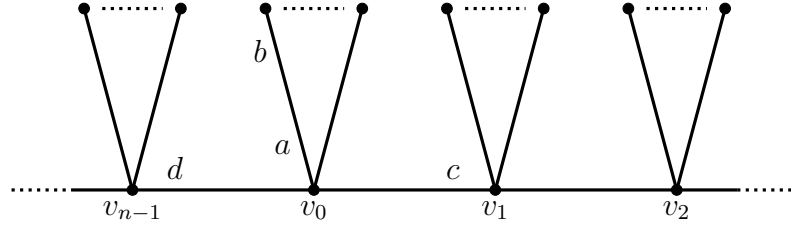
Figure 10: Colouring the external incidences of v_1 ($v_{n-1} = v_2$ if $n = 3$), pre-coloured case.

For every such i, we denote by α_i the colour assigned to one or both external incidences of v_i. If we are in the pre-coloured case, we first deal with the external incidences of v_1 and v_{n-1}.

- *External incidences of v_1, pre-coloured case* (see Figure 10).
  Let $L'(v_0, v_0v_1) = L(v_0, v_0v_1) \setminus \{a, b, c, d\}$, and

  $$L'(v_2, v_2v_1) = \begin{cases} L(v_2, v_2v_1) \setminus \{c, d\}, & \text{if } n = 3, \\ L(v_2, v_2v_1) \setminus \{c\}, & \text{otherwise.} \end{cases}$$

  If $L'(v_0, v_0v_1) \cap L'(v_2, v_2v_1) \neq \emptyset$, then we set $\sigma_0(v_0, v_0v_1) = \sigma_0(v_2, v_2v_1) = \alpha_1$ for some $\alpha_1 \in L'(v_0, v_0v_1) \cap L'(v_2, v_2v_1)$. Otherwise, we consider two cases.
    - If $n = 3$, then $|L(v, vu)| \geq \max(p + 3, 8)$ for every incidence (v, vu) of $C_n \odot pK_1$, which implies $|L'(v_0, v_0v_1)| \geq \max(p-1, 4)$ and $|L'(v_2, v_2v_1)| \geq \max(p+1, 6)$, so that $|L'(v_0, v_0v_1) \cup L'(v_2, v_2v_1)| \geq \max(2p, 10)$. Therefore, either there exists some colour $\alpha_1 \in L'(v_0, v_0v_1) \setminus L(v_1, v_1v_1^p)$, in which case we set $\sigma_0(v_0, v_0v_1) = \alpha_1$, or there exists some colour $\alpha_1 \in L'(v_2, v_2v_1) \setminus L(v_1, v_1v_1^p)$, and we set $\sigma_0(v_2, v_2v_1) = \alpha_1$.
    - If $n \geq 4$, then $|L(v, vu)| \geq \max(p + 3, 7)$ for every incidence (v, vu) of $C_n \odot pK_1$, which implies $|L'(v_0, v_0v_1)| \geq \max(p-1, 3)$ and $|L'(v_2, v_2v_1)| \geq \max(p + 2, 6)$, so that $|L'(v_0, v_0v_1) \cup L'(v_2, v_2v_1)| \geq \max(2p + 1, 9)$. Therefore, either there exists some colour $\alpha_1 \in L'(v_0, v_0v_1) \setminus L(v_1, v_1v_1^p)$, in which case we set $\sigma_0(v_0, v_0v_1) = \alpha_1$, or there exists some colour $\alpha_1 \in L'(v_2, v_2v_1) \setminus L(v_1, v_1v_1^p)$, and we set $\sigma_0(v_2, v_2v_1) = \alpha_1$.
- *External incidences of v_{n-1}, pre-coloured case.*
  Let $L'(v_0, v_0v_{n-1}) = L(v_0, v_0v_{n-1}) \setminus \{a, b, c, d, \alpha_1\}$, and

  $$L'(v_{n-2}, v_{n-2}v_{n-1}) = \begin{cases} L(v_{n-2}, v_{n-2}v_{n-1}) \setminus \{c, d, \alpha_1\}, & \text{if } n = 3, \\ L(v_{n-2}, v_{n-2}v_{n-1}) \setminus \{d, \alpha_1\}, & \text{if } n = 4, \\ L(v_{n-2}, v_{n-2}v_{n-1}) \setminus \{d\}, & \text{otherwise.} \end{cases}$$

  If $d \notin L(v_{n-1}, v_{n-1}v_{n-1}^p)$, then we set $\sigma_0(v_0, v_0v_{n-1}) = \alpha_{n-1}$ for some $\alpha_{n-1} \in L'(v_0, v_0v_{n-1})$ and we are done.
  Suppose now that $d \in L(v_{n-1}, v_{n-1}v_{n-1}^p)$. If $L'(v_{n-2}, v_{n-2}v_{n-1}) \cap L'(v_0, v_0v_{n-1}) \neq \emptyset$, then we set $\sigma_0(v_{n-2}, v_{n-2}v_{n-1}) = \alpha_{n-1}$ and $\sigma_0(v_0, v_0v_{n-1}) = \alpha_{n-1}$ for some $\alpha_{n-1} \in L'(v_{n-2}, v_{n-2}v_{n-1}) \cap L'(v_0, v_0v_{n-1})$. Otherwise, we consider two cases.
    - If $n = 3$ (and thus, $(v_{n-2}, v_{n-2}v_{n-1}) = (v_1, v_1v_2)$), then $|L(v, vu)| \geq \max(p + 3, 8)$ for every incidence (v, vu) of $C_3 \odot pK_1$, which implies
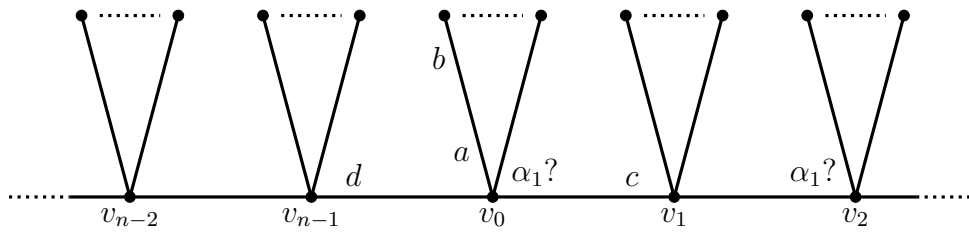
Figure 11: Colouring the external incidences of v_{n-1} ($v_{n-2} = v_2$ if $n = 4$), pre-coloured case. At least one of the incidences (v_0, v_0v_1) or (v_2, v_2v_1) is coloured with α_1.

$|L'(v_1, v_1v_2)| \geq \max(p, 5)$ and $|L'(v_0, v_0v_2)| \geq \max(p - 2, 3)$, so that $|L'(v_1, v_1v_2) \cup L'(v_0, v_0v_2)| \geq \max(2p - 2, 8)$. Note that $L'(v_1, v_1v_2) \cup L'(v_0, v_0v_2) \neq L(v_2, v_2v_2^p)$ since $d \in L(v_2, v_2v_2^p)$ and $d \notin L'(v_1, v_1v_2) \cup L'(v_0, v_0v_2)$. Therefore, either there exists some colour $\alpha_2 \in L'(v_1, v_1v_2) \setminus L(v_2, v_2v_2^p)$, in which case we set $\sigma_0(v_1, v_1v_2) = \alpha_2$, or there exists some colour $\alpha_2 \in L'(v_0, v_0v_2) \setminus L(v_2, v_2v_2^p)$, and we set $\sigma_0(v_0, v_0v_2) = \alpha_2$.

– If $n \geq 4$ (see Figure 11), then $|L(v, vu)| \geq \max(p + 3, 7)$ for every incidence (v, vu) of $C_n \odot pK_1$, which implies $|L'(v_{n-2}, v_{n-2}v_{n-1})| \geq \max(p + 1, 5)$ and $|L'(v_0, v_0v_{n-1})| \geq \max(p - 2, 2)$, so that $|L'(v_{n-2}, v_{n-2}v_{n-1}) \cup L'(v_0, v_0v_{n-1})| \geq \max(2p-1, 7)$. As in the previous case, $L'(v_{n-2}, v_{n-2}v_{n-1})) \cup L'(v_0, v_0v_{n-1}) \neq L(v_{n-1}, v_{n-1}v_{n-1}^p)$ since $d \in L(v_{n-1}, v_{n-1}v_{n-1}^p)$ and $d \notin L'(v_{n-2}, v_{n-2}v_{n-1}) \cup L'(v_0, v_0v_{n-1})$. Therefore, either there exists some colour $\alpha_{n-1} \in L'(v_{n-2}, v_{n-2}v_{n-1}) \setminus L(v_{n-1}, v_{n-1}v_{n-1}^p)$, in which case we set $\sigma_0(v_{n-2}, v_{n-2}v_{n-1}) = \alpha_{n-1}$, or there exists some colour $\alpha_{n-1} \in L'(v_0, v_0v_{n-1}) \setminus L(v_{n-1}, v_{n-1}v_{n-1}^p)$, and we set $\sigma_0(v_0, v_0v_{n-1}) = \alpha_{n-1}$.

For constructing the partial colouring σ_0, we proceed sequentially, from $i = 2$ to $i = n - 2$ if we are in the pre-coloured case and $n \neq 3$ (note that σ_0 is already constructed if $n = 3$), or from $i = 0$ to $i = n - 1$ otherwise. For each such i, let

$$L'(v_{i-1}, v_{i-1}v_i) = L(v_{i-1}, v_{i-1}v_i) \setminus \{\alpha_{i-2}, \alpha_{i-1}, \alpha_{i+1}\}, \text{ and}$$

$$L'(v_{i+1}, v_{i+1}v_i) = L(v_{i+1}, v_{i+1}v_i) \setminus \{\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+2}\},$$

if we are not in the pre-coloured case, or

$$L'(v_{i-1}, v_{i-1}v_i) = \begin{cases} L(v_{i-1}, v_{i-1}v_i) \setminus \{c, \alpha_1, \alpha_3\}, & \text{if } i = 2, \\ L(v_{i-1}, v_{i-1}v_i) \setminus \{\alpha_{i-2}, \alpha_{i-1}, \alpha_{i+1}\}, & \text{otherwise,} \end{cases} \text{ and}$$

$$L'(v_{i+1}, v_{i+1}v_i) = \begin{cases} L(v_{i+1}, v_{i+1}v_i) \setminus \{\alpha_{n-3}, \alpha_{n-1}, d\}, & \text{if } i = n - 2, \\ L(v_{i+1}, v_{i+1}v_i) \setminus \{\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+2}\}, & \text{otherwise,} \end{cases}$$

if we are in the pre-coloured case.

Note here that when proceeding with i, the colour α_{i-2} (resp. $\alpha_{i-1}, \alpha_{i+1}, \alpha_{i+2}$) is defined only if $i \geq 2$ (resp. $i \geq 1$, $i \leq n - 1$, $i \leq n - 2$).

If $L'(v_{i-1}, v_{i-1}v_i) \cap L'(v_{i+1}, v_{i+1}v_i) \neq \emptyset$, we set $\sigma_0(v_{i-1}, v_{i-1}v_i) = \sigma_0(v_{i+1}, v_{i+1}v_i) = \alpha_i$ for some $\alpha_i \in L'(v_{i-1}, v_{i-1}v_i) \cap L'(v_{i+1}, v_{i+1}v_i)$.

Otherwise, since $|L(v, vu)| \geq \max(p + 3, 7)$ for every incidence (v, vu) of $C_n \odot pK_1$, which implies $|L'(v_{i-1}, v_{i-1}v_i)| \geq \max(p, 4)$ and $|L'(v_{i+1}, v_{i+1}v_i)| \geq \max(p, 4)$, so that $|L'(v_{i-1}, v_{i-1}v_i) \cup L'(v_{i+1}, v_{i+1}v_i)| \geq \max(2p, 8)$, either there exists some colour $\alpha_i \in L'(v_{i-1}, v_{i-1}v_i) \setminus L(v_i, v_iv_i^p)$, in which case we set $\alpha_0(v_{i-1}, v_{i-1}v_i) = \alpha_i$, or there exists some colour $\alpha_i \in L'(v_{i+1}, v_{i+1}v_i) \setminus L(v_i, v_iv_i^p)$, and we set $\alpha_0(v_{i+1}, v_{i+1}v_i) = \alpha_i$.

By construction, the partial L-list incidence colouring σ_0 clearly satisfies Property (P').

(b) We now colour the remaining uncoloured incidences of C_n, which can be done since every such incidence has at most four already coloured adjacent incidences. Thanks to Property (P'), and since at least one of the external incidences of each vertex v_i has been coloured in the previous step, the partial L-list incidence colouring σ_0 thus obtained satisfies Property (P).

2. We now extend the partial L-list incidence colouring σ_0 to an L-list incidence colouring σ of $C_n \odot pK_1$. The only remaining uncoloured incidences are the internal and external incidences of pendent vertices (except $(v_0, v_0v_0^1)$ and $(v_0^1, v_0^1v_0)$ if we are in the pre-coloured case, which are already coloured by a and b, respectively).

We proceed as follows. If we are in the pre-coloured case, then we first colour the incidences $(v_0, v_0v_0^2)$, ..., $(v_0, v_0v_0^p)$, in that order, otherwise we first colour the incidences $(v_0, v_0v_0^1)$, ..., $(v_0, v_0v_0^p)$, in that order. Then, for each vertex v_i, $1 \leq i \leq n - 1$, we colour the incidences $(v_i, v_iv_i^1)$, ..., $(v_i, v_iv_i^p)$, in that order. This can be done since, doing so,

(a) every incidence $(v_i, v_iv_i^j)$, $1 \leq j \leq p - 1$, has $j + 3 \leq p + 2$ already coloured adjacent incidences (recall that $|L(v_i, v_iv_i^j)| \geq p + 3$), and

(b) thanks to Property (P) (and to the fact that $|L(v_0, v_0v_0^p) \cap \{a, b, c, d\}| \leq 2$ if we are in the precoloured case), the incidence $(v_i, v_iv_i^p)$ has at most $p+2$ forbidden colours.

We finally colour all the uncoloured incidences of the form $(v_i^j, v_i^jv_i)$, $0 \leq i \leq n - 1$, $1 \leq j \leq p$, which can be done since every such incidence has $p + 2$ already coloured adjacent incidences.

This completes the proof. $\square$

We now able to prove the main result of this section. Let G be a cactus, and C be a cycle in G. We say that C is a *maximal cycle* if C contains a vertex v with $\deg_G(v) = \Delta(G)$.

Theorem 16 *Let G be a cactus which is neither a tree nor a cycle. We then have*

$$\mathrm{ch}_i(G) \leq \begin{cases} \Delta(G) + 2, & \text{if } \Delta(G) = 3, \\ \Delta(G) + 1, & \text{if } \Delta(G) = 4 \text{ and } G \text{ has no maximal cycle}, \\ \Delta(G) + 2, & \text{if } \Delta(G) = 4 \text{ and } G \text{ has a maximal cycle}, \\ \max(\Delta(G) + 1, 7), & \text{if } \Delta(G) \geq 5 \text{ and } G \text{ has at most one maximal 3-cycle}, \\ \max(\Delta(G) + 1, 8), & \text{otherwise}. \end{cases}$$

Proof. Let L be a list assignment of G such that $|L(v, vu)| = k$ for every incidence (v, vu) of G, where k is the value claimed in the statement of the theorem.

Let $C_1, \ldots, C_\ell$, $\ell \geq 1$, denote the cycles in G, and M denote the graph obtained from G by contracting each cycle C_i into a vertex c_i. The graph M is clearly a tree. Let us call each vertex c_i in M a *cycle vertex* and each other vertex in M, if any, a *normal vertex*. Moreover, if G contains a maximal 3-cycle, we assume without loss of generality that this cycle is C_1. We now order all the vertices of M, starting with c_1, in such a way that each vertex $v \neq c_1$ has exactly one neighbour among the vertices preceding v in the order (this can be done since M is a tree).

We now colour the incidences of G according to the ordering of the vertices of M as follows. Let v be the vertex of M to be treated. We have two cases to consider.

1. *v is a cycle vertex of M.*
   Let $v = c_i$, $1 \leq i \leq \ell$. We then colour all the incidences of the subgraph H_i of G induced by the vertices of the cycle C_i and their neighbours. The subgraph H_i is a subgraph of some generalized corona and thus, thanks to Observation 1 and Lemma 15, all the incidences of H_i can be coloured.

2. *v is a normal vertex of M.*
   In that case, v is also a vertex in G. We colour the uncoloured internal incidences of v, if any, and then the uncoloured external incidences of v, if any, in that order. This can be done since, doing so, every such incidence has at most $\Delta(G)$ already coloured adjacent incidences.

This concludes the proof. □

Note that thanks to Proposition 3, the bound given in Theorem 16 is tight for every cactus G such that $\Delta(G) \geq 7$, or $\Delta(G) = 6$ and G has at most one maximal 3-cycle, or $\Delta(G) = 4$ and G has no maximal cycle.

6   Hamiltonian cubic graphs

By Proposition 4, we know that $\mathrm{ch}_i(G) \leq 7$ for every graph with maximum degree 3. We prove in this section that this bound can be decreased to 6 for Hamiltonian cubic graphs. (Recall that by the result of Maydanskyi [10], $\chi_i(G) \leq 5$ for every cubic graph.)

Let G be a Hamiltonian cubic graph of order n (n is necessarily even) and $C_G = v_0v_1 \ldots v_{n-1}v_0$ be a Hamilton cycle in G. The set of edges $F = E(G) \setminus E(C_G)$ is thus a perfect matching. We denote by F_G the subgraph of G induced by F. Let v_i, $0 \le i \le n-1$, be a vertex of G. The *matched vertex* of v_i (with respect to C_G) is the unique vertex v_j such that $v_iv_j \in F$. The *antipodal vertex* of v_i (with respect to C_G) is the vertex $v_{i+\frac{n}{2}}$ (subscripts are taken modulo n). Two vertices x and y of G are *consecutive* (with respect to C_G) if there exists some i, $0 \le i \le n-1$, such that $\{x, y\} = \{v_i, v_{i+1}\}$ (subscripts are taken modulo n).

We first prove the following easy lemma.

Lemma 17 *If G is a Hamiltonian cubic graph of order $n \ge 6$ and $C_G = v_0v_1 \ldots v_{n-1}v_0$ a Hamilton cycle in G, then there exists a vertex v_i in G, $0 \le i \le n-1$, such that v_{i+2} is not the matched vertex of v_i.*

Proof. If v_2 is not the matched vertex of v_0 then v_0 satisfies the required property. Otherwise, since $n \ge 6$, v_2 satisfies the required property. $\qquad\square$

We now prove the main result of this section.

Theorem 18 *For every Hamiltonian cubic graph G, $\mathrm{ch}_i(G) \le 6$.*

Proof. Let G be a Hamiltonian cubic graph, $C_G = v_0v_1 \ldots v_{n-1}v_0$ be a Hamilton cycle in G, and L be any list assignment of G such that $|L(v, vu)| = 6$ for every incidence (v, vu) of G. In the following, subscripts are always taken modulo n.

Note first that if $n = 4$, then $G = K_4$ and the result follows from Lemma 13. We thus assume $n \ge 6$. Each vertex v_i, $0 \le i \le n-1$, has three neighbours in G, namely v_{i-1}, v_{i+1} and the matched vertex $v_i' = v_j$ of v_i, $j \in \{0, \ldots, n\} \setminus \{i-1, i, i+1\}$. Let v_s and v_t denote the matched vertices of v_0 and v_1, respectively. Without loss of generality, we may assume that v_0 satisfies the statement of Lemma 17, so that $v_s \ne v_2$.

The following claim will be useful for constructing an L-list incidence colouring of G.

Claim 3 *There exist $a \in L(v_1, v_1v_t)$, $b \in L(v_s, v_sv_0)$, $c \in L(v_2, v_2v_1)$, $d \in L(v_0, v_0v_s)$ and $e \in L(v_t, v_tv_1)$, with $a \ne c$, $a \ne e$ and $b \ne d$, such that*

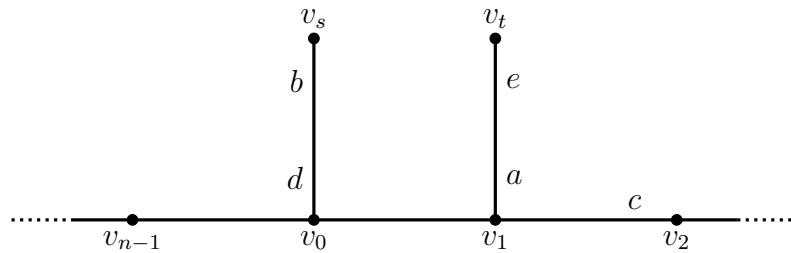$$|L(v_0, v_0v_1) \cap \{a, b\}| \le 1, \quad and \quad |L(v_1, v_1v_0) \cap \{c, d, e\}| \le 1.$$

Figure 12: Configuration for the proof of Claim 3.

Proof. We first deal with the incidence (v_1, v_1v_0) and set the values of c, d and e (see Figure 12). Let $C = L(v_2, v_2v_1)$, $D = L(v_0, v_0v_s)$ and $E = L(v_t, v_tv_1)$. If $C \cap D \cap E \neq \emptyset$, then we set $c = d = e = \gamma$ for some $\gamma \in C \cap D \cap E$, so that $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$.

Otherwise, we consider two cases.

1. If C, D and E are pairwise disjoint, then at least two of them are distinct from $L(v_1, v_1v_0)$, so that we can choose c, d and e in such a way that $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$.

2. Suppose now that $C \cap D \neq \emptyset$ (the cases $C \cap E \neq \emptyset$ and $D \cap E \neq \emptyset$ are similar). We first set $c = d = \gamma$ for some $\gamma \in C \cap D$. If $\gamma \in L(v_1, v_1v_0)$, then there exists $\varepsilon \in E \setminus L(v_1, v_1v_0)$ (since $(C \cap D) \cap E = \emptyset$) and we set $e = \varepsilon$, so that $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$. If $\gamma \notin L(v_1, v_1v_0)$, then we set $e = \varepsilon$ for any $\varepsilon \in E$ and we also get $|L(v_1, v_1v_0) \cap \{c, d, e\}| \leq 1$.

We now deal with the incidence (v_0, v_0v_1) and set the values of a and b. Let $L'(v_1, v_1v_t) = L(v_1, v_1v_t) \setminus \{e, c\}$ and $L'(v_s, v_sv_0) = L(v_s, v_sv_0) \setminus \{d\}$. If $L'(v_1, v_1v_t) \cap L'(v_s, v_sv_0) \neq \emptyset$, then we set $a = b = \alpha$ for some $\alpha \in L'(v_1, v_1v_t) \cap L'(v_s, v_sv_0)$, so that $|L(v_0, v_0v_1) \cap \{a, b\}| \leq 1$. Otherwise, as $|L(v, vu)| = 6$ for every incidence (v, vu) of G, which implies $|L'(v_1, v_1v_t)| \geq 4$ and $|L'(v_s, v_sv_0)| \geq 5$, we get $|L'(v_1, v_1v_t) \cup L'(v_s, v_sv_0)| \geq 9$. Therefore, either there exists some colour $\alpha \in L'(v_s, v_sv_0) \setminus L(v_0, v_0v_1)$, in which case we set $b = \alpha$, so that $|L(v_0, v_0v_1) \cap \{a, b\}| \leq 1$ for any value of a, or there exists some colour $\alpha \in L'(v_1, v_1v_t) \setminus L(v_0, v_0v_1)$, in which case we set $a = \alpha$, so that $|L(v_0, v_0v_1) \cap \{a, b\}| \leq 1$ for any value of b. This completes the proof of Claim 3. $\square$

We now construct an L-list incidence colouring σ of G in three steps.

1. We first set $\sigma(v_1, v_1v_t) = a$, $\sigma(v_s, v_sv_0) = b$, $\sigma(v_2, v_2v_1) = c$, $\sigma(v_0, v_0v_s) = d$ and $\sigma(v_t, v_tv_1) = e$, where a, b, c, d and e are the values determined in the proof of Claim 3.

2. We colour all the uncoloured incidences of the perfect matching $F = E(G) \setminus E(C_G)$. This can be done since every such incidence has at most two already coloured adjacent incidences (indeed, only the lastly coloured incidence of the edge $v_2 v_2'$, where v_2' is the antipodal vertex of v_2, will have two already coloured adjacent incidences).

3. We finally colour all the uncoloured incidences of C_G (the only incidence of C_G already coloured is $(v_2, v_2 v_1)$) as follows.

   - We first colour the incidence $(v_1, v_1 v_2)$, which has four already coloured adjacent incidences.

   - We then cyclically colour the incidences of C_G from $(v_2, v_2 v_3)$ to $(v_0, v_0 v_{n-1})$. This can be done since, doing so, every such incidence has four or five already coloured adjacent incidences.

   - By Claim 3, the incidence $(v_0, v_0 v_1)$ has at most five forbidden colours and can thus be coloured. Similarly, thanks to Claim 3, the incidence $(v_1, v_1 v_0)$ has at most five forbidden colours and can thus be coloured.

This completes the proof of Theorem 18.   □

By Observation 1, we get the following corollary of Theorem 18.

Corollary 19 *If G is a Hamiltonian graph with maximum degree 3, then $\mathrm{ch}_i(G) \leq 6$.*

7   Discussion

In this paper, we have introduced and studied the list version of incidence colouring. We determined the exact value of – or upper bounds on – the incidence choice number of several classes of graphs, namely square grids, Halin graphs, generalized coronae of cycles, cactuses and Hamiltonian cubic graphs. Following the work presented here, we propose the following problems:

1. Is it true that $\mathrm{ch}_i(G_{m,n}) = 6$ for every square grid $G_{m,n}$ with $m \geq n \geq 3$?

2. What is the best possible upper bound on the list incidence chromatic number of Halin graphs with maximum degree 3, 4 or 5? (Theorem 14 gives the exact bound only for Halin graphs with maximum degree $k \geq 6$.)

3. What is the best possible upper bound on the list incidence chromatic number of cactuses with maximum degree 6? with maximum degree 5 and containing at most one maximal cycle? with maximum degree 4 and containing a maximal cycle? [Theorem 16 gives the exact bound for all other cases.)

4. What is the best possible upper bound on the list incidence chromatic number of graphs with bounded maximum degree? In particular, what about graphs with maximum degree 3? (By Proposition 4, we know that this bound is at most $3k-2$ for graphs with maximum degree $k \geq 2$, and thus at most 8 for graphs with maximum degree 3.)

5. What is the value of $\mathrm{ch}_i(K_n)$? (By Proposition 4, we know that this value is at most $3n - 5$.)

6. Which classes of graphs satisfy the incidence version of the list colouring conjecture, that is, for which graphs G do we have $\mathrm{ch}_i(G) = \chi_i(G)$? (By Proposition 3 and Theorem 6, we know for instance that this equality holds for every tree.)

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