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Enumeration of Incongruent Quadrilaterals with Distinct Integer Sides and Fixed Perimeter

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Abstract : In this paper, we present a new explicit formula for counting incongruent quadrilaterals with distinct integer side lengths and a fixed perimeter n. Our method integrates combinatorial techniques and integer partition properties to address this classical problem in geometry. We prove the validity of the formula and offer computational examples for different perimeter values. This work builds on prior results in triangle enumeration and adds to the wider field of discrete geometry.

Keywords: Incongruent quadrilaterals, Integer partitions, Polygonal partition, Generating function.

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1 Introduction and background

In 1977, J. H. Jordan, R. Walch, and R. J. Wisner [5] investigated the properties of triangles with integer side lengths, developing methods to count triangles with a fixed perimeter and integer sides through generating functions and combinatorial techniques. Two decades later, in 1997, Andrews examined the enumeration of such triangles under specific constraints, linking this problem to integer partition theory [1]. His findings established a formula relating the count of these triangles to the partition function $p(n)$, which represents the number of ways an integer n can be expressed as a sum of positive integers, irrespective of order (see, e.g., [2, 4, 7]). In 2019, James East and Ron Niles tackled a related problem, focusing on enumerating distinct integer m-gons with a given perimeter n, using dihedral group actions on $\mathcal{D}_n$ [6]. This paper addresses the problem of counting incongruent quadrilaterals with distinct side lengths for a given perimeter n. Bouroubi has explored similar problems, specifically for incongruent ordered integer quadrilaterals [3].

Theorem 1 ([3]) *The number of incongruent ordered integer quadrilaterals with perimeter $n \geq 4$ is:*

$$\left\{ \frac{1}{576} n \left(n+3\right) \left(2n+3\right) - \frac{\left(-1\right)^n}{192} n \left(n-5\right) \right\},$$

where, $\{x\}$ denotes the nearest integer to the real number x.

2 Definitions and notations

- A partition of a positive integer n is a way of representing n as a sum of positive integers. Formally, a partition of n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that:

  $$0 < \lambda_1 \leq \cdots \leq \lambda_k \quad \text{and} \quad \lambda_1 + \cdots + \lambda_k = n.$$

  Here, $\lambda_1, \ldots, \lambda_k$ are called the parts of the partition, and k denotes the length of the partition (i.e., the number of parts).

  Two partitions that differ only in the order of their parts are considered identical. For example, the integer 5 has the following partitions:

  $$(5);\ (1,4);\ (2,3);\ (1,1,3);\ (1,2,2);\ (1,1,1,2);\ (1,1,1,1,1).$$

- A partition λ of a positive integer n into k distinct parts is defined as:

  $$0 < \lambda_1 < \cdots < \lambda_k.$$

- A partition λ is said to be polygonal if it satisfies the polygonal inequality, that is:

  $$\lambda_1 + \cdots + \lambda_{k-1} > \lambda_k.$$

  If also, the parts are all distinct, the partition is called a polygonal partition with distinct parts.

- The following notations, used in this paper, provide a way to distinguish between different types of partitions based on the number and properties of the parts:

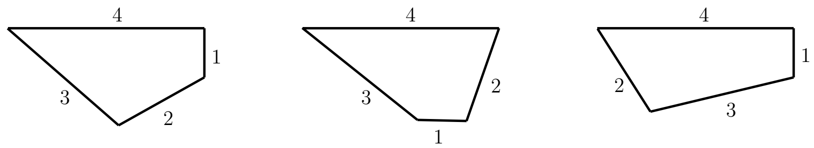
  - $p(n, k)$: the number of partitions of n into exactly k parts.
  - $q(n, k)$: the number of partitions of n into exactly k distinct parts.
  - $q^g(n, k)$: the number of polygonal partitions of n into k distinct parts.
  - $q^{\overline{g}}(n, k)$: the number of non-polygonal partitions of n into k distinct parts.

  Of course, we have:
  $$q(n, k) = q^g(n, k) + q^{\overline{g}}(n, k).$$

- Let us define congruence between two k-gons P and Q as follows: Let P and Q be two k-gons with side lengths $\lambda_1, \ldots, \lambda_k$ and $\mu_1, \ldots, \mu_k$, respectively. We consider P and Q to be congruent if and only if the k-tuple $(\lambda_1, \ldots, \lambda_k)$ can be transformed into $(\mu_1, \ldots, \mu_k)$ through a combination of cyclic reordering and/or reversal of the sequence, regardless of the starting side and the direction of reading (clockwise or counterclockwise).

  - The number of incongruent k-gons with perimeter n, having distinct ordered side lengths, is denoted by $g(n, k)$.
  - The number of incongruent k-gons with perimeter n, having distinct side lengths in any order, is denoted by $\mathcal{G}(n, k)$.

  For $n = 10$, there exist 3 incongruent quadrilaterals with distinct side lengths, among which only one is ordered, namely the leftmost one.



3 Auxiliary results

Lemma 2 *For $n \geq 10$ and $k \geq 3$, it holds that:*

$$q^{\overline{g}}(n, k) = \sum_{m=6}^{\lfloor \frac{n}{2} \rfloor} q(m, k-1).$$

Here, $\lfloor x \rfloor$ denotes the floor function, which gives the greatest integer less than or equal to the real number x.

Proof. A non-polygonal partitions λ of n in k distinct parts, $\lambda_1, \ldots, \lambda_k$, is solution of system:

$$\begin{cases} n = \lambda_1 + \cdots + \lambda_{k-1} + \lambda_k; \\ 0 < \lambda_1 < \cdots < \lambda_k, \\ \lambda_k \geq \lambda_1 + \cdots + \lambda_{k-1}. \end{cases} \tag{1}$$

Thus, we have

$$n - \lambda_k \leq \lambda_k.$$

Therefore,

$$\lambda_k \geq \frac{n}{2}\cdot$$

This leads to the conclusion that

$$n - \lambda_k \leq \frac{n}{2}\cdot$$

Consequently, the number of solutions to the system (1) is equal to the number of partitions of $n - \lambda_k$ into $k-1$ parts, where $n - \lambda_k \leq \lfloor \frac{n}{2} \rfloor$. ■

For the specific case where $n = 18$ and $k = 4$, there exist seven non-polygonal partitions, which can be enumerated as follows:

$(1, 2, 3, 12)$; $(1, 2, 4, 11)$; $(1, 3, 4, 10)$; $(1, 2, 5, 10)$; $(1, 2, 6, 9)$; $(1, 3, 5, 9)$ and $(2, 3, 4, 9)$.

Which is clearly confirmed by our result:

$$q^{\overline{g}}(18, 4) = \sum_{m=6}^{\lfloor 9 \rfloor} q(m, k-1) = 1 + 1 + 2 + 3.$$

Observation 3 *For $n \geq 10$ and $k \geq 3$, we can also state that:*

$$q^{\overline{g}}(n, k) = \sum_{m=3}^{\lfloor \frac{n}{2} \rfloor} p\left(m - \binom{k-1}{2}, k-1\right),$$

where $\binom{i}{j}$ is the binomial coefficient.

The result follows from the following identity [1]:

$$q(n, k) = p\left(n - \frac{k(k-1)}{2}, k\right).$$

4 Main results

Theorem 4 *For $n \geq 10$ and $k \geq 3$, it follows that:*

$$g(n, k) = q(n, k) - \sum_{m=6}^{\lfloor \frac{n}{2} \rfloor} q(m, k-1).$$

Proof. Given a polygonal partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with distinct parts, we can construct a polygon with k sides of distinct lengths corresponding to the parts of the partition, and conversely, each such polygon corresponds to a polygonal partition with distinct parts.

Thus, we have

$$g(n,k) = q^g(n,k).$$

Leveraging the relation $q(n,k) = q^g(n,k) + q^{\overline{g}}(n,k)$, the result immediately follows from Lemma 2. ■

When $n = 18$, we observe 8 polygonal partitions into 4 parts, namely:

$(1,4,5,8)$; $(1,3,6,8)$; $(1,2,7,8)$; $(2,3,5,8)$; $(2,4,5,7)$; $(1,4,6,7)$; $(2,3,6,7)$ and $(2,4,5,7)$.

This aligns with our theoretical result, as demonstrated by the equation:

$$g(18,4) = q(18,4) - \sum_{m=6}^{9} q(m,3) = 15 - 7 = 8.$$

Theorem 5 *For $n \geq 6$, we have*

$$\sum_{m=6}^{n} q(m,3) = \frac{n^3}{36} - \frac{5n^2}{24} + \frac{11n}{36} + \frac{1}{3}\left\lfloor \frac{n}{3} \right\rfloor + \frac{(-1)^n - 1}{16}.$$

Proof. The generating function of $q(n,3)$ is given by [2]:

$$f(z) = \frac{z^6}{(1-z)(1-z^2)(1-z^3)}.$$

Then

$$\sum_{m=6}^{n} q(m,3) = [z^n]\left(\frac{f(z)}{1-z}\right).$$

By expressing $\frac{f(z)}{1-z}$ as a sum of partial fractions, we obtain:

$$\frac{f(z)}{1-z} = \frac{1}{9}\frac{z+1}{(z^2+z+1)} + \frac{119}{144}\frac{1}{(z-1)} + \frac{1}{16}\frac{1}{(z+1)} + \frac{89}{72}\frac{1}{(z-1)^2} + \frac{3}{4}\frac{1}{(z-1)^3} + \frac{1}{6}\frac{1}{(z-1)^4}.$$

After some straightforward calculations, we find:

$$\frac{z+1}{z^2+z+1} = \sum_{n \geq 0} a_n z^n,$$

where

$$a_n = 1 - n + 3\left\lfloor \frac{n}{3} \right\rfloor.$$

Consequently, the sum becomes

$$\sum_{m=6}^{n} q(m,3) = \frac{n^3}{36} - \frac{5n^2}{24} + \frac{11n}{36} + \frac{1}{3}\left\lfloor \frac{n}{3} \right\rfloor + \frac{(-1)^n - 1}{16}.$$

■

Corollary 6 *For $n \geq 12$, we have:*

$$\sum_{m=6}^{\left\lfloor \frac{n}{2} \right\rfloor} q(m,3) = \frac{1}{576} n\,(n-11)\,(2n+3\,(-1)^n-11)+\frac{5}{48}(-1)^n+\frac{1}{3}\left\lfloor \frac{2n+(-1)^n-1}{12} \right\rfloor + \\ +\frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}}{16}-\frac{1}{6}.$$

Proof. The result follows simply through substitution and straightforward computations from Theorem 5, by considering that:

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2} - \frac{1-(-1)^n}{4}.$$

■

Theorem 7 *For $n \geq 10$, the number of partitions of n into exactly k distinct parts is:*

$$q(n,4) = \frac{n^3}{144}-\frac{5n^2}{48}+\frac{(9\,(-1)^n+103)\,n}{288} \\ +\frac{1}{96}\left(32\left\lfloor \frac{n-1}{3} \right\rfloor - 6\,(-1)^{\left\lfloor \frac{3n}{2} \right\rfloor} - 6(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} - 15(-1)^n - 37\right).$$

Proof. The generating function for $q(n,4)$ is given as follows [2]:

$$g(z) = \frac{z^{10}}{(1-z)\,(1-z^2)\,(1-z^3)\,(1-z^4)}.$$

By expanding $g(z)$ into partial fractions, we obtain:

$$g(z) = 1+\frac{19}{16\,(z-1)}-\frac{3}{16\,(z+1)}+\frac{239}{288\,(z-1)^2}+\frac{1}{32\,(z+1)^2}+\frac{7}{24\,(z-1)^3}+\frac{1}{24\,(z-1)^4} \\ -\frac{1}{8\,(z^2+1)}-\frac{1}{9\,(z^2+z+1)}.$$

Through expansions in power series and detailed calculations, we get:

$$g(z) = 1+\sum_{n\geq 0}\left(-\frac{19}{16}-\frac{3\,(-1)^n}{16}+\frac{239\,(n+1)}{288}+\frac{(-1)^n\,(n+1)}{32}-\frac{7\,(n+1)\,(n+2)}{48}\right) \\ +\sum_{n\geq 0}\left(\frac{(n+1)\,(n+2)\,(n+3)}{144}-\frac{b_n}{8}-\frac{c_n}{9}\right) z^n,$$

where

$$b_n = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor}\left(\frac{1+(-1)^n}{2}\right),$$

and

$$c_n = n - 2 - 3\left\lfloor \frac{n-1}{3} \right\rfloor.$$

Direct computation verifies the validity of the result. ■

Theorem 8 *For $n \geq 10$, the number of incongruent quadrilaterals with perimeter n and distinct ordered side lengths is given by:*

$$g(n,4) = \left\{ \frac{n^3}{288} - \frac{\left((-1)^n + 9\right)n^2}{192} + \frac{17\left(3(-1)^n + 5\right)n}{576} - \frac{25(-1)^n}{96} - \frac{1}{3}\left\lfloor \frac{2n + (-1)^n - 1}{12} \right\rfloor + \frac{1}{3}\left\lfloor \frac{n-1}{3} \right\rfloor \right\}.$$

Proof. By applying Theorem 4, Corollary 6, and Theorem 7, and substituting k with 3, we obtain:

$$\begin{aligned} g(n,4) &= \frac{n^3}{288} - \frac{\left((-1)^n + 9\right)n^2}{192} + \frac{17\left(3(-1)^n + 5\right)n}{576} - \frac{25(-1)^n}{96} - \frac{1}{3}\left\lfloor \frac{2n + (-1)^n - 1}{12} \right\rfloor \\ &\quad + \frac{1}{3}\left\lfloor \frac{n-1}{3} \right\rfloor - \frac{1}{8}(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} - \frac{1}{16}(-1)^{\left\lfloor \frac{3n}{2} \right\rfloor} - \frac{7}{32}. \end{aligned}$$

The result is confirmed, as the following inequality holds:

$$\left| -\frac{1}{8}(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} - \frac{1}{16}(-1)^{\left\lfloor \frac{3n}{2} \right\rfloor} - \frac{7}{32} \right| < \frac{1}{2}.$$

■ Finally, we present the principal result of this paper, which summarizes the key findings of our study.

Theorem 9 *For $n \geq 10$, the number of incongruent quadrilaterals with perimeter n and distinct side lengths, whether in ordered or unordered form, is given by:*

$$\mathcal{G}(n,4) = 3g(n,4).$$

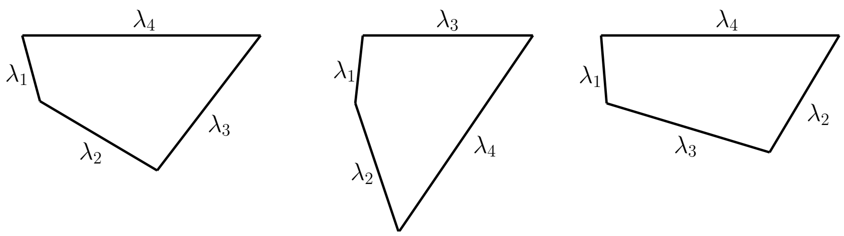
Which implies:

$$\mathcal{G}(n,4) = 3\left\{ \frac{n^3}{288} - \frac{\left((-1)^n + 9\right)n^2}{192} + \frac{17\left(3(-1)^n + 5\right)n}{576} - \frac{25(-1)^n}{96} - \frac{1}{3}\left\lfloor \frac{2n + (-1)^n - 1}{12} \right\rfloor + \frac{1}{3}\left\lfloor \frac{n-1}{3} \right\rfloor \right\}.$$

Proof. Consider a polygonal partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of n into 4 distinct parts, satisfying:

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \quad \text{and} \quad n = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4.$$

This partition uniquely determines an ordered quadrilateral with perimeter n. Furthermore, each partition gives rise to two additional incongruent quadrilaterals, as shown in the figure below:



■

5 Numerical Application

The following table gives calculated a few values of $\mathcal{G}(n,4)$, for $10 \leq n \leq 30$.

n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\mathcal{G}(n,4)$	3	3	3	6	9	12	15	21	24	33	36	48	54	69	72	93	99	123	129	159	165

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