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A new upper bound for the coalition number

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Abstract : A *coalition partition* of a graph $G = (V, E)$ is a partition of its vertex-set into $k \geq 1$ subsets $V_1, V_2, \dots, V_k$ such that each subset V_i is either (i) a singleton dominating set or (ii) not a dominating set but $V_i \cup V_j$ forms a dominating set for some other subset V_j. Such a partition is called a c-partition. The *coalition number* of a graph G, denoted $C(G)$, is the largest number of subsets in a c-partition of G. In this paper, we establish a new upper bound for $C(G)$ and characterize all triangle-free graphs achieving this bound.

Keywords: Coalition; Coalition partition; Coalition number.

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1 Introduction

In this paper, we consider only graphs that are finite, undirected and simple. Let $G = (V, E)$ be a graph of order $n = |V|$. The *complement* of a graph $G = (V, E)$ is a graph $\overline{G}$ defined on the same vertex set V, where two vertices are adjacent in $\overline{G}$ if and only if are not adjacent in G. For a non-empty set $A \subseteq V$, we denote by $G \backslash A$ the subgraph induced by $V \backslash A$. For a vertex v of G, the *degree* of v is the number of edges incident to v. A vertex of degree $n-1$ is called a *full vertex*, while a vertex with degree zero is called an *isolated* vertex. If all vertices of G are isolated, we call G an *empty graph*, and we denoted it by $\overline{K}_n$. The *distance* between two vertices u and v in a connected graph G is the length of the shortest path between them. The *diameter,* denoted $\mathrm{diam}(G)$ of a graph G is the maximum distance between any two vertices in G. The *union* of two vertex-disjoint graphs G and H is the graph $G+H$ whose vertex-set is $V(G) \cup V(H)$ and edge-set is $E(G) \cup E(H)$. For a given graph H, a graph G is called H-free if G does not contain H as an induced subgraph. A *bipartite* graph G is a graph whose vertex-set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in G connects a vertex in V_1 to a vertex in V_2. A vertex v in V_1 (respectively, in V_2) is called a *charismatic vertex* if it is adjacent to every vertex in V_2 (respectively, in V_1). If all vertices in V_1 and V_2 are charismatic, G is a *complete bipartite* and is denoted by $K_{p,q}$, where $|V_1| = p$ and $|V_2| = q$. As usual, K_n, P_n and C_n denote the complete graph, path and cycle on n vertices, respectively. For other notation and terminology not defined here we refer the reader to [8, 21].

A set $D \subseteq V$ in a graph G is called a *dominating set* of G if every vertex not in D has at least one neighbor in D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A dominating set of G with cardinality $\gamma(G)$ is called a γ-set of G. The concept of the domination in graphs has been studied extensively and several research papers have been published on this topic. For a survey on this area, we refer the reader to [12].

For a given graph G with vertex set V, two subsets V_1 and V_2 of V are said to form a *coalition* in G if neither V_1 nor V_2 is a dominating set, but their union $V_1 \cup V_2$ is a dominating set. A *coalition partition* (c-partition for short) in G is a partition of its vertex-set into k subsets $V_1, V_2, \ldots, V_k$ such that every set V_i is either a singleton dominating set, or is not a dominating set but forms a coalition with another set V_j $(j \neq i)$.

The *coalition number* of a graph G, denoted by $C(G)$, is the maximum k such that G admits a c-partition of cardianlity k. A c-partition of G with $k = C(G)$ is called a $C(G)$-partition.

The concept of c-partition was first introduced and studied by Haynes et al. in [13]. They proved that every graph G admits a c-partition and, as a consequence, deduced that every graph G of order $n \geq 1$ satisfies the following:

$$1 \leq C(G) \leq n. \tag{1}$$

They also determined the exact coalition numbers for paths and cycles in [13]. The same authors defined in [15] the notion of *coalition graph* and provided additional results in [14, 16, 17]. Further studies have explored the c-partition problem in certain specific classes of graphs. For instance, Bakhshesh et al. [4, 6] in trees and Alikhani et al. [1, 9]

in cubic graphs. Other results regarding coalition partition for other parameters have been undertaken by Alikhani et al. for total coalition in [3] and for connected coalition in [2, 11]; Samadzadeh et al. for independent coalition in [22] and for paired coalition in [23]; Jafari et al. [18] for k-coalition; Mojdeh et al. for perfect coalition in [20] and for edge coalition in [19]; Golmohammadi et al. [10] for strong coalition. Further works on this topic can be found in [5, 7].

The main purpose of this paper is to propose a new upper bound for $C(G)$ and characterize all triangle-free graphs that attain this bound.

2 Preliminary Results

We start this section by giving an upper bound for the coalition number $C(G)$ in terms of n and $\gamma(G)$.

Theorem 1 *Let G be a graph with order n and domination number γ. Then*

$$C(G) \leq n - \gamma(G) + 2. \tag{2}$$

This bound is sharp.

Proof. For the case $\gamma(G) = 1$, the result is obvious from (1). So, assume that $\gamma(G) \geq 2$. Therefore G has no full vertex. Set $k = C(G)$ and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a $C(G)$-partition of G. Then, we can write

$$n = |V_1| + |V_2| + \cdots + |V_k|\,. \tag{3}$$

Without loss of generality, assume that V_1 and V_2 form a coalition. Then $|V_1|+|V_2| \geq \gamma(G)$. Combining this with (3), we obtain

$$n \geq |V_1| + |V_2| + k - 2 \geq \gamma(G) + k - 2. \tag{4}$$

This completes the proof. ■

The inequality (2) is sharp, for instance, for $\overline{K}_n$, for $K_p + K_{n-p}$ and for $K_p + \overline{K}_{n-p}$ (with $2 \leq p \leq n-1$), although this is not the case for complete graphs and stars. As $\gamma(G) \geq p$ when G has p components, the following result is immediate from Theorem 1.

Corollary 2 *If G is a graph of order n with $p \geq 2$ connected components, then*

$$C(G) \leq n - p + 2.$$

Proposition 3 *Let G be a graph of order n, diameter* diam(G), *domination number γ and with p connected components. If $C(G) = n$, then the following properties hold.*

(i) $\gamma(G) \leq 2$.

(ii) $p \leq 2$, *with equality if and only if G is the disjoint union of two complete graphs.*

(iii) If G is connected, then $\operatorname{diam}(G) \leq 3$.

Proof. (i) Follows from (2).

(ii) If $p \geq 3$, then $\gamma(G) \geq 3$, contradicting (i). Assume now that $p = 2$ and let G_1 and G_2 be the two components of G. For each i in $\{1, 2\}$, let v_i be any vertex in G_i. Since $C(G) = n$, $\{v_1\}$ form a coalition with some singleton set in G_2, say $\{v_2\}$. In this case, v_i must be adjacent to all the other vertices in G_i. Thus G_1 and G_2 both are complete graphs.

iii) Suppose that $d \geq 4$ and assume that v_0-v_1-$\cdots$-v_d be a diametral path in G. In this case, the set $\{v_2\}$ cannot form a coalition with any other singleton set, which contradicts that $C(G) = n$. ■

3 Main result

Our aim result is the following.

Theorem 4 *If G is a K_3-free graph with at least two vertices, then equality holds in (2) if and only if $G \in \mathcal{G} \cup \mathcal{H} \cup \{C_5, K_2 + \overline{K}_p\ (p \geq 1),\ \overline{K}_n\ (n \geq 2)\}$.*

The proof of Theorem 4 relies on the following definitions and lemmas.

Definition 1 (Family $\mathcal{G}$) *A graph G is in class $\mathcal{G}$ if it is obtained from $p \geq 1$ disjoint stars, each with at least three vertices, by adding a new vertex and connecting it to all the leaves of the stars, and possibly adding some isolated vertices. When G has no isolated vertices, it must hold that $p \geq 2$.*

Definition 2 (Family $\mathcal{H}$) *A graph G is in class $\mathcal{H}$ if its vertex set can be partitioned into two disjoint classes X_1 and X_2 such that:*

- *G is bipartite with bipartition (X_1, X_2), where $|X_1|, |X_2| \geq 2$.*
- *Every vertex in X_1 (respectively, X_2) has at most one non-neighbor in X_2 (respectively, X_1).*
- *If $|X_1| \neq |X_2|$, then both X_1 and X_2 contain at least one charismatic vertex.*

Remark that every member G of $\mathcal{H}$ is connected unless G is $2K_2$. In addition, G has no full vertex, implying that $\gamma(G) \geq 2$. Since, there exists a vertex u in X_1 and a vertex v in X_2 such that u and v together dominate all vertices in G, it follows that $\gamma(G) = 2$.

The Figure 1 shows an example of three graphs : one is in the family $\mathcal{G}$ and the others are in the family $\mathcal{H}$, along with their domination numbers.

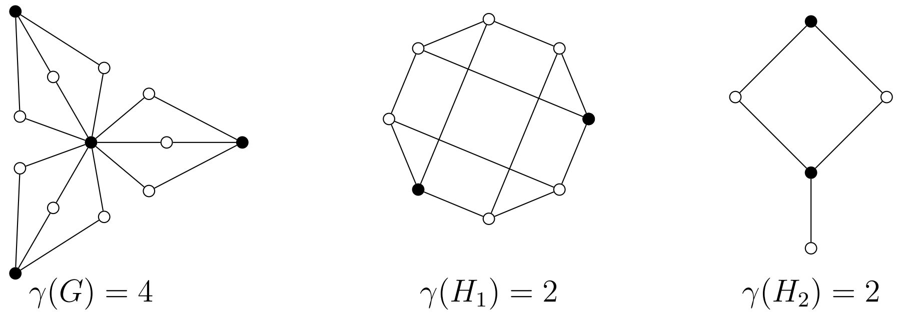

Figure 1: Example of three graphs $G \in \mathcal{G}$ and $H_1, H_2 \in \mathcal{H}$ with their domination numbers.

Lemma 5 *Let G be a graph of order $n \geq 2$ and domination number $\gamma(G)$. If G is a member of $\mathcal{G} \cup \mathcal{H}$, then $C(G) = n - \gamma(G) + 2$.*

Proof. From(2), it suffices to show that $C(G) \geq n - \gamma(G) + 2$. If $G \in \mathcal{H}$, then it is straightforward to check that the partition of $V(G)$ into n singleton substes is a c-partition of G. Thus $C(G) \geq n = n - \gamma(G) + 2$ (since $\gamma(G) = 2$ by the remark before Lemma 5).

Assume now $G \in \mathcal{G}$. Let D be a γ-set of G, and let I (possibly empty) denote the set of isolated vertices in G, with $q = |I|$. Clearly $I \subseteq D$. For each $i \in \{1, 2, \ldots, p\}$, let c_i be the center of the i-th star, and let u be the vertex adjacent to all leaves of the $p \geq 1$ stars. By definition of $\mathcal{G}$, we have $p + q \geq 2$. We first show that $|D| = p + q + 1$. Indeed, to dominate all vertices of G, the minimality of D requires that D must include all center vertices together with I and exactly one vertex among $V(G) \backslash (I \cup \{c_1, c_2, \ldots, c_p\})$. This imples that $|D| \geq p + q + 1$. To establish equality, we construct a dominating set D of size $p + q + 1$ by taking all vertices of I, all the center vertices of the stars, and u. Thus $|D| \leq p + q + 1$ implying that

$$|D| = \gamma(G) = p + q + 1. \tag{5}$$

Now, let $S = \{c_1, c_2, \ldots, c_p\} \cup I$ and let $v_1, v_2, \ldots, v_{n-(p+q)}$ be the vertices of $V(G) \backslash S$. It is easy to check that $\{S, \{v_1\}, \{v_2\}, \ldots, \{v_{n-(p+q)}\}\}$ is a c-partition of G with cardinality $n - (p + q) + 1$. Therefore $C(G) \geq n - (p + q) + 1$. Combining this with (5), we get the desired result.

In both cases, we have shown that $C(G) = n - \gamma(G) + 2$. ■

Recall that a *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle.

Lemma 6 *Let G be a graph. If $\gamma(G) = 2$, then G contains no odd cycle of length greater than* 5.

Proof. Suppose for the sake of contraction that G contains an odd cycle $C : v_1\text{-}v_2\text{-}\cdots\text{-}v_{2q+1}\text{-}v_1$ (in this order) with $q \geq 3$. Assume that C has minimum length among all odd cycles of G. Observe that C has no chords, as otherwise, G would contain a smaller odd cycle C' such that $V(C') \subset V(C)$, which contradicts the choice of C. Let $D = \{x, y\}$ be a γ-set of G, and consider the following two cases.

Case 1. One of x and y, say x lies on C.

Without loss of generality, we can let $x = v_1$. Since C has no chords, the neighbors of x on C are exactly v_2 and v_{2q+1}. Next, we consider two possibilities, depending on whether y lies on C or not.

Possibility 1. $y \in C$. Similarly, y is adjacent to exactly two vertices of C. Thus D can dominate at most 6 vertices on C (two vertices dominated by x and two by y, plus x and y themselves). However, C contains $2q+1 \geq 7$ vertices, meaning that D cannot dominate all vertices of C, a contradiction.

Possibilty 2. $y \notin C$. Since D is a γ-set of G, y must dominate all the remaining vertices of C, in particular v_3 and v_4. But then $\{y, v_3, v_4\}$ induces a triangle, a contradiction.

Case 2. Neither x nor y lies on C.

Since D is a γ-set of G and C is odd, by the pigeonhole principle, one vertex in D, say x must be adjacent to two consecutive vertices of C, meaning that x is adjacent to v_i and v_{i+1} for some i in $\{1, 2, \ldots, 2q + 1\}$. But in this case, $\{x, v_i, v_{i+1}\}$ induces a triangle, a contradiction.

In both cases, we have a contradiction and thus the required is done. ■

We are now ready to prove Theorem 4.

Proof of Theorem 4. The sufficiency is immediate for $G \in \{C_5, K_2 + \overline{K}_p\ (p \geq 1), \overline{K}_n\ (n \geq 2)\}$. Furthermore, if G is a member of $\mathcal{G} \cup \mathcal{H}$, it follows directly from Lemma 5. To establish the necessity, let G be traingle-free graph of order $n \geq 2$ and set

$$k = n - \gamma(G) + 2. \tag{6}$$

If $k \leq 3$, then (6) yields $\gamma(G) \in \{n, n-1\}$ implying that G is isomorphic to $\overline{K}_n$ or $K_2 + \overline{K}_{n-2}$ $(n \geq 3)$. So, assume that $k \geq 4$ and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a c-partition of G. If $\gamma(G) = 1$, then (6) becomes $k = n + 1$, contradicting (1). Thus $\gamma(G) \geq 2$. Based on this, we consider two cases.

Case 1. $\gamma(G) \geq 3$.

Then $n = k + \gamma(G) - 2 \geq 5$. Assume that $|V_1| \geq |V_i|$ for all $i \geq 2$ and define two subsets A and B of $V(G)$ as follows:

- $A = \{x \in V(G) \backslash V_1 : x \text{ has a neighbor in } V_1\}$,
- $B = V(G) \backslash (V_1 \cup A)$.

Clearly V_1, A and B are pairwise disjoint sets and $V(G) = V_1 \cup A \cup B$. Let I (possibly empty) be the set of isolated vertices in G.

Claim 1.

(i) $|V_i| = 1$ for each $i \neq 1$, and $|V_1| = \gamma(G) - 1 \geq 2$.

(ii) For all $i \geq 2$, V_i must necessarily form a coalition with V_1.

(iii) $|B| = 1$, A is a nonempty independent and further $A \cup B$ induces a star centered at the unique vertex of B. As consequence, $I \subseteq V_1$.

(iv) V_1 is an independent that contains I as a subset and $V_1 \backslash I \neq \emptyset$.

(v) Every vertex in $V_1 \backslash I$ has at least two neighbors in A, and every vertex in A has exactly one neighbor in $V_1 \backslash I$.

Proof of Claim 1. (i) Assume without loss of generality that V_1 forms a coalition with V_2. Clearly,

$$|V_1| + |V_2| \geq \gamma(G). \tag{7}$$

Suppose to the contrary that for some $i_0 \geq 3$, $|V_{i_0}| \geq 2$. Using (7), we get:

$$|V_1| + |V_2| + |V_{i_0}| \geq \gamma(G) + 2, \tag{8}$$

which, combined with (3) implies that $n \geq \gamma(G) + k - 1$, Contradicting (6). Hence,

$$|V_i| = 1 \text{ for each } i \geq 3. \tag{9}$$

It remains to show that $|V_2| = 1$ and $|V_1| = \gamma(G) - 1$. Indeed, by (9) and (3), we have:

$$n = |V_1| + |V_2| + k - 2. \tag{10}$$

Substituting k with $n - \gamma(G) + 2$ into (10), we get

$$|V_1| + |V_2| = \gamma(G). \tag{11}$$

Suppose to the contrary $|V_2| \geq 2$. Then by the choice of V_1 and (11),

$$|V_2| \leq |V_1| \leq \gamma(G) - 2. \tag{12}$$

Taking into account (9) together with (12) and the fact that $\gamma(G) \geq 3$, we see that the set V_i (for $i \geq 3$) does not form a coalition with any other set in π, a contradiction. Thus $|V_2| = 1$ and by (11), we have $|V_1| = \gamma(G) - 1$. As $\gamma(G) \geq 3$, it follows that $|V_1| \geq 2$.

(ii) Directly follows from (i) as $\gamma(G) \geq 3$.

(iii) $B \neq \emptyset$, for otherwise V_1 forms a dominating set with cardinality $\gamma(G) - 1$, a contradiction. Thus, pick $b \in B$. By (ii), $\{b\}$ must form a coaltion with V_1. Therefore, since there is no edge between V_1 and B, it follows that $\{b\}$ must dominate all the other vertices of B. Combined with G being triangle-free, this implies that B is a clique with at most two vertices.

In view of (i), we know that $|\pi| = k = 1 + |A| + |B|$. As $|B| \leq 2$ and $k \geq 4$, it follows that $A \neq \emptyset$. Let $a \in A$. Using reasoning similar to that for b, we deduce that $\{a\}$ dominates all vertices of B. Hence there are all possible edges between A and B. If $|B| \geq 2$, then two adjacent vertices in B together with some vertex in A would induce a triangle, contradicting the triangle-free property of G. Similarly, if A is not independent, then two adjacent vertices in A together with some vertex in B induce a triangle. Consequently, $A \cup B$ induces a star centered at the unique vertex in B.

iv) From (i) and (iii), we can let $v \in V_1$ and $B = \{b\}$. If v has a neighbor in V_1, then, by considering (i) and (ii), the set $(V_1 \backslash \{v\}) \cup \{b\}$ is a dominating set of cardinality $\gamma(G)-1$, a contradiction. Hence V_1 is an independent set. The definition of A implies that $V_1 \backslash I \neq \emptyset$.

v) As V_1 is independent, every vertex in $V_1 \backslash I$ must have at least one neighbor in A. Let $a \in A$ be a neighbor of v in A . If a is the only neighbor of v in A, then, by taking (i) and (ii) into consideration, we see that the set $(V_1 \backslash \{v\}) \cup \{a\}$ is a dominating set of cardinality $\gamma(G) - 1$, a contradiction. Thus every vertex in $V_1 \backslash I$ must have at least two neighbors in A, which implies $|A| \geq 2$.
Suppose now that a has another neighbor in $V_1 \backslash I$, say $v' \neq v$. In this case, $(V_1 \backslash \{v, v'\}) \cup \{a, b\}$ forms a dominating set of cardinality $\gamma(G) - 1$, a contradiction again. Thus every vertex in A has exactly one neighbor in $V_1 \backslash I$. This finishes the proof of Claim 1.

By Claim 1, we see that $|V_1| \geq 2$, $|A| \geq 2$ and the subgraph induced by $(V_1 \backslash I) \cup A$ consists of $p \geq 1$ stars, each contains at least three vertices, with a center in $V_1 \backslash I$ and leaves in A. Furtheremore, the set B contains a single vertex that is adjacent to all the vertices in A. From this, we conclude that $G \in \mathcal{G}$.

Case 2. $\gamma(G) = 2$.
Then from (6), it follows that $k = n \geq 4$, implying that each set in π is a singleton. If G is disconnected, Proposition 3-(ii) and the fact that G is triangle-free with $n \geq 4$ yield $G = 2K_2 \in \mathcal{H}$. Now, asume that G is connected. If $G = C_5$, we are done. Thus, we may assume that $G \neq C_5$. We assert that

$$G \text{ is bipartite.} \tag{13}$$

Suppose not, and let C be the shortest odd cycle in G, with vertex-set $V(C) = \{v_1, v_2, \ldots, v_t\}$ and edge-set $E(C) = \{v_1v_2, v_2v_3, \ldots, v_{t-1}v_t, v_tv_1\}$. By Lemma 6 and the triangle-free property of G, it follows that $t = 5$ and C is an induced cycle. Since $G \neq C_5$ and G is connected, there exists a vertex $u \in V(G) \backslash V(C)$ that is adjacent to some vertex in C, say v_1. As G is triangle-free, u cannot be adjacent to v_2, v_5 and to one of v_3, v_4 (assume v_3 without loss of generality). Now, consider a set $\{w\}$ in π that forms a coalition with $\{u\}$. Such a set must dominate v_2, v_5 and v_3. However, the set $\{w, v_2, v_3\}$ induces a triangle, a contradiction. Thus (13) holds.

By (13), we can write $V(G) = X_1 \cup X_2$, where X_1 and X_2 are the two parts of G. As G is connected and has no full vertex, it follows that $|X_1| \geq 2$ and $|X_2| \geq 2$.

Claim 2. Every vertex in X_1 has at most one non-neighbor in X_2 and vice versa. Moreover, if $X_1 \neq X_2$, then for each $i \in \{1, 2\}$, X_i contains at least one charismatic vertex.

Proof of Claim 2. Suppose on the contrary that there exists $j \in \{1, 2\}$ such that X_j contains a vertex having two non-neighbors $u, v \in X_{3-j}$. In this case neither $\{u\}$ nor $\{v\}$

can form a coalition with any other set in π, a contradiction. To prove the second part, assume without loss of generality that $|X_1| > |X_2|$. For each $i \in \{1,2\}$, define

$$Y_i = \{v \in X_i : v \text{ has exactly one non-neighbor in } X_{3-i}\}.$$

Clearly $|Y_1| = |Y_2|$ and therefore $X_1 \backslash Y_1 \neq \emptyset$. Hence X_1 contains at least one charismatic vertex. If $X_2 \backslash Y_2 = \emptyset$, then any vertex in $X_1 \backslash Y_1$ cannot form a coalition with any other set in π, a contradiction. Thus $X_2 \backslash Y_2 \neq \emptyset$ implying that X_2 contains at least one vertex charismatic. This conclude the proof of Claim 2.

It follows from our preceding discussions that G is a member of the family $\mathcal{H}$. This ends the proof of Theorem 4. ■

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