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Kaplansky's Formula Revisited

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Abstract : This work is devoted to the study of linear and circular k-separated subsets. Revisiting Kaplansky's classical formula, we establish new perspectives through refined bijective arguments and alternative proofs that shed light on the structural properties of these families. In addition, we derive several new combinatorial identities that both generalize and encompass previously known results. Some of these identities arise naturally from recurrence relations, while others are obtained via structural decompositions. Taken together, these results not only extend Kaplansky's framework but also provide a unified approach to the enumeration of k-separated subsets, thereby opening new directions for further research in algebraic and enumerative combinatorics.

Keywords: k-separated subsets ; Kaplansky's formula ; combinatorial identities ; bijective proofs ; recurrence relations.

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1 Introduction and background

Enumerative problems involving separated elements on linear and circular arrangements have been a source of continued interest in combinatorics. The most fundamental case, which forbids direct adjacency between selected elements, was first solved by Kaplansky in 1943 [3]. He provided exact formulas for the number of ways to choose k elements from n such that no two are consecutive, both for arrangements on a line, $\binom{n-k+1}{k}$, and on a circle, $\frac{n}{n-k}\binom{n-k}{k}$ [3]. A more intricate set of constraints was later studied by Mansour and Sun [5].

A further generalization, also considered by Kaplansky [4], involves selecting k-subsets $\{x_1, \ldots, x_k\}$ from $\mathbb{Z}_n$. The condition is that for any two distinct x_i, x_j, their difference modulo n cannot lie in $\{1, 2, \ldots, s\}$, namely:

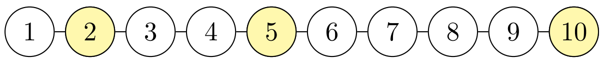
$$x_i - x_j \not\equiv 1, 2, \ldots, s \pmod{n}.$$

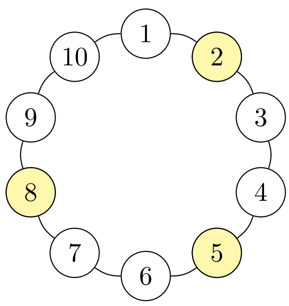
Under the prerequisite that $n \geq ks + 1$, Kaplansky showed that the count of these subsets is:

$$\frac{n}{n-ks}\binom{n-ks}{k}.$$

2 s-separated subsets

Let $E = \{1, 2, \ldots, n\}$ be a finite set of integers. A subset $A = \{x_1, x_2, \ldots, x_k\}$ of E, with its elements arranged in ascending order $1 \leq x_1 < x_2 < \cdots < x_k \leq n$, is said to be *$s$-separated* if the difference between any two consecutive elements is at least $s + 1$. We distinguish two categories of s-separated subsets, namely the linear and the circular types, illustrated below for the case $n = 10$, $k = 3$, and $s = 2$:





Notation 1 *Throughout this article, $f_s(n, k)$ and $g_s(n, k)$ denote the numbers of linear and circular s-separated subsets of $\{1, 2, \ldots, n\}$, respectively. In accordance with Kaplansky's notation, the parameter s will be omitted when $s = 1$, i.e., we write $f(n, k)$ and $g(n, k)$.*

Graph-theoretic interpretation. The enumeration of s-separated subsets finds a natural counterpart in graph theory, specifically as a problem of counting independent sets. An independent set, by definition, is a collection of vertices in a graph where no two vertices are adjacent. This connection is most apparent in the base case where $s = 1$. Here, selecting a 1-separated subset of size k from $\{1, 2, \dots, n\}$ is precisely equivalent to choosing an independent set of k vertices in the path graph P_n.

This framework readily generalizes to an arbitrary integer s, where the separation condition translates to a distance constraint on the graph: any two selected vertices must be separated by a distance of at least $s + 1$. This interpretation extends naturally to the circular arrangement by replacing the path graph P_n with the cycle graph C_n.

Thus, the enumeration of s-separated subsets corresponds to counting independent sets of a fixed cardinality in graphs derived from paths and cycles, where the adjacency condition is extended to a minimum distance requirement.

2.1 Linear s-separated subset

The condition for a linear s-separated subset can be formalized as follows:

$$x_1 \geq 1,\ x_i - x_{i-1} \geq s + 1\ (2 \leq i \leq k),\ \text{and } x_n \leq n.$$

To solve this constrained counting problem, we employ a bijective transformation. Define a sequence a sequence $(y_i)_{1 \leq i \leq k+1}$ by the relations:

$$y_1 = x_1 - 1, y_i = x_i - x_{i-1} - (s + 1),\ \forall i \in \{1, \dots, k\} \text{ and } y_{k+1} = n - x_n.$$

Then we have

$$\begin{cases} y_1 + y_2 + \cdots + y_{k+1} = n - 1 - (k - 1)(s + 1), \\ y_i \geq 0,\ \forall i \in \{1, \dots, k + 1\}. \end{cases} \tag{1}$$

By the classical *stars principal*, the number of such solutions $\{y_1, y_2, \dots, y_{k+1}\}$ is given by the binomial coefficient:

$$\binom{n - (k - 1)s}{k}.$$

Therefore, we obtain

$$f_s(n, k) = \binom{n - (k - 1)s}{k}. \tag{2}$$

Based on a double-counting argument, we can rederive $f_s(n, k)$ by partitioning the solutions of System (1) according to the value of y_{k+1}.

Fix $y_{k+1} = j$. The problem then reduces to counting the number of non-negative integer k-tuples $(y_1, \dots, y_k)$ satisfying:

$$y_1 + y_2 + \cdots + y_k = n - 1 - (k - 1)(s + 1) - j. \tag{3}$$

For a fixed j, the number of such solutions is given by the binomial coefficient:

$$\binom{n - (k - 1)s - j - 1}{k - 1}.$$

Summing over all admissible values of j, namely $0 \leq j \leq n-1-(k-1)(s+1)$, yields the total. Finally, reindexing the sum via the change of variables $i = j+1$ gives the identity:

$$f_s(n,k) = \sum_{i=1}^{n-(k-1)(s+1)} \binom{n-(k-1)s-i}{k-1}. \tag{4}$$

The immediate consequence of this identity is the well-known hockey-stick identity [1, 6, 2]:

$$\binom{N}{r+1} = \sum_{i=1}^{N-r} \binom{N-i}{r},$$

by setting $r = k-1$, $N = n-(k-1)s$.

2.2 Generalized recurrence relation

Kaplansky [3] proved the following relation for $s = 1$:

$$f(n,k) = f(n-1,k) + f(n-2,k-1).$$

Building on Kaplansky's approach [3], we derive a recurrence for the general s-adjacent case, by partitioning the set of valid subsets based on the inclusion of the element n. We consider two disjoint cases:

1. **The subset does not contain n.** In this case, we must choose k s-separated elements from the set $\{1, 2, \ldots, n-1\}$. The number of ways to do so is $f_s(n-1,k)$.
2. **The subset contains n.** If n is selected, the s-separation rule forbids choosing any element from $\{n-s, \ldots, n-1\}$. The remaining $k-1$ elements must therefore be chosen from the set $\{1, 2, \ldots, n-s-1\}$. The number of ways is $f_s(n-s-1, k-1)$.

By the sum rule, adding the counts from these two cases yields the desired recurrence:

$$f_s(n,k) = f_s(n-1,k) + f_s(n-s-1,k-1).$$

2.3 Circular s-separated subset

A cornerstone in the study of k-separated subsets is the classical result of Kaplansky [3].

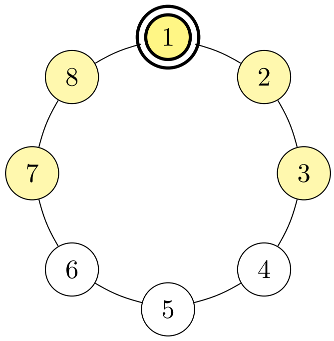
Proposition 1 (Kaplansky) *Let $n, k \in \mathbb{N}$ with $n \geq k(s+1)$. The number of circular k-separated subsets of size k in $\{1, 2, \ldots, n\}$ is given by:*

$$\frac{n}{n-ks}\binom{n-ks}{k}.$$

Despite the classical status of Proposition 1, we present a new proof based on a bijective construction. This approach not only offers an alternative argument but also sheds light on the underlying structure, serving as a foundation for several generalizations presented in later sections.

Bijective proof.

Let $G_s(n,k)$ and $F_s(n,k)$ denote the sets of circular and linear s-separated k-subsets of $E = \{1,2,\ldots,n\}$, respectively. For an element $a \in E$, the *s-neighborhood* centered at a, denoted $V_s(a)$, is the set of $2s+1$ elements $\{a-s,\ldots,a-1,a,a+1,\ldots,a+s\}$ (operations are performed modulo n in the circular case). For instance, the figure bellow illustrates the 2-neighborhood centered at 1 in $E = \{1,2,\ldots,8\}$.



Let us considere the two sets :

$$A = \{(S,a) \mid S \in G_s(n,k),\ a \in S\},$$

and

$$B = \{(a,S) \mid a \in E,\ S \in F_s(n \setminus V_s(a),\ k-1)\}.$$

Define the mapping

$$\varphi : A \longrightarrow B, \quad \varphi(S,a) = (a,\ S \setminus V_s(a)).$$

By construction φ is bijective. Consequently,

$$|A| = |B| \implies k\, g_s(n,k) = n\, f_s(n-2s-1,\ k-1).$$

Applying formula (2), we deduce that:

$$g_s(n,k) = \frac{n}{k}\binom{n-sk-1}{k-1}.$$

Equivalently,

$$g_s(n,k) = \frac{n}{k} \cdot \frac{k}{n-sk}\binom{n-sk}{k}$$

$$= \frac{n}{n-sk}\binom{n-sk}{k}.$$

This completes the proof. ■

Although the bijective argument is enlightening, an algebraic approach yields deeper insight into the underlying structure. We therefore present an algebraic proof.

Algebraic proof.

In the circular setting, besides the constraints already imposed in the linear case, the separation condition must also hold between the largest and the smallest elements of $A = \{x_1, x_2, \ldots, x_k\}$. Formally, the conditions can be stated as:

$$x_1 \geq 1,\ x_i - x_{i-1} \geq s+1\ (2 \leq i \leq k),\ x_k \leq n,\ \text{and}\ n - x_k + x_1 \geq s+1.$$

These inequalities guarantee that, in the cyclic arrangement, there are at least s elements of E separating any two consecutive members of A.

To count these subsets, we transform the problem by defining non-negative variables $(z_i)_{1 \leq i \leq k+1}$ that represent the sizes of the "excess" gaps between elements:

$$z_1 = x_1 - 1,\ z_i = x_i - x_{i-1} - (s+1)\ \text{for}\ 2 \leq i \leq k,\ \text{and}\ z_{k+1} = n - x_k.$$

The problem's constraints imply that $z_i \geq 0$ for all i. Summing these variables reveals a constant total:

$$z_1 + z_2 + \cdots + z_{k+1} = n - 1 - (k-1)(s+1).$$

Furthermore, the wrap-around condition $n - x_k + x_1 \geq s+1$ can be rewritten in terms of our new variables as $(n - x_k) + (x_1 - 1) \geq s$, which is simply $z_{k+1} + z_1 \geq s$. We are therefore looking for the number of non-negative integer solutions to the system:

$$\begin{cases} z_1 + z_2 + \cdots + z_{k+1} = n - 1 - (k-1)(s+1), \\ z_{k+1} + z_1 - s \geq 0. \end{cases} \tag{5}$$

Setting $z_{k+1} + z_1 = l$, System (5) can be rewritten as:

$$\begin{cases} z_2 + \cdots + z_k = n - 1 - (k-1)(s+1) - l, \\ s \leq l \leq n - 1 - (k-1)(s+1). \end{cases} \tag{6}$$

Since $z_{k+1} + z_1 = l$ admits $l+1$ solutions, we get:

$$g_s(n,k) = \sum_{l=s}^{n-1-(k-1)(s+1)} (l+1) \binom{n-2-(k-1)\,s-l}{k-2} \tag{7}$$

Set $i = l+1$. Then Equation (7) becomes:

$$g_s(n,k) = \sum_{i=s+1}^{n-(k-1)(s+1)} i \binom{n-(k-1)\,s-(i+1)}{k-2}. \tag{8}$$

This settles the proof. ∎

Now we are able to formulate our result as follows:

Theorem 2 *The following identity holds:*

$$\frac{n}{n-ks}\binom{n-ks}{k} = \sum_{i=s+1}^{n-(k-1)(s+1)} i \binom{n-(k-1)\,s-(i+1)}{k-2}. \tag{9}$$

A direct consequence of Theorem 2 arises in the case $s = 0$. After simplification, this yields the identity:

$$\binom{n+1}{k+2} = \sum_{i=1}^{n-k} i \binom{n-i}{k}.$$

We recognize this as an instance of the well-known binomial convolution identity [1],

$$\sum_{i=a}^{n-b} \binom{i}{a}\binom{n-i}{b} = \binom{n+1}{a+b+1}, \quad a+b \le n,$$

corresponding to the specific parameters $a = 1$ and $b = k$.

3 Recurrence formula for $f_s(n,k) - g_s(n,k)$

A central objective of this section is to provide a formula for the difference $f_s(n,k) - g_s(n,k)$, given the constraint $f_s(n,k) > g_s(n,k)$. We shall establish a recurrence relation that governs this value, the proof of which relies on a combinatorial argument using the Principle of Inclusion-Exclusion.

Theorem 3 *The difference between $f_s(n,k)$ and $g_s(n,k)$ is given by:*

$$f_s(n,k) - g_s(n,k) = \sum_{i=1}^{s} i\, f_s(n - 2s - i - 1,\ k-2).$$

Proof. Cutting the circle between 1 and n and unfolding it transforms any generalized circular k-separated subset into a generalized linear k-separated subset. However, the converse does not hold: the missing k-subsets $(x_1, \dots, x_k)$ are precisely those satisfying:

$$(x_1 - 1) + (n - x_k) \le s - 1.$$

For each equation

$$(x_1 - 1) + (n - x_k) = j, \quad j = 0, \dots, s-1,$$

there are exactly $(j+1)\, f_s(n-2s-j-2,\ k-2)$ generalized linear k-separated subsets that must be excluded. The multiplicative factor $(j+1)$ arises from the number of nonnegative integer solutions to $j_1 + j_2 = j, \quad j_1, j_2 \ge 0$, when setting $x_1 - 1 = j_1$ and $n - x_k = j_2$. ■

The next result is an immediate consequence of Theorem 3 obtained by setting $s = 1$. It coincides with Kaplansky's Matching Rounds Theorem [3].

Corollary 4 *For all n and $k \in \mathbb{N}^*$, $n - 2k \ge 0$, we have:*

$$g(n,k) = f(n,k) - f(n-4, k-2).$$

4 Conclusion

This study reaffirms the foundational role of Kaplansky's framework in the enumeration of k-separated subsets, while also demonstrating that its scope can be significantly broadened through bijective reasoning and structural decompositions. The novel combinatorial identities we have established illustrate how classical results can be unified and extended within a cohesive algebraic and enumerative setting. This approach not only elucidates the internal structure of k-separated families but also highlights the potential for such methods to intersect with broader areas of discrete mathematics. We anticipate that the perspectives outlined here-ranging from asymptotic analyses to applications in graph theory and optimization-will stimulate further developments and deepen the understanding of separation phenomena in combinatorics.

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